MA3111 Complex Analysis I

Notes by Chan Heng Huat

Contents

References

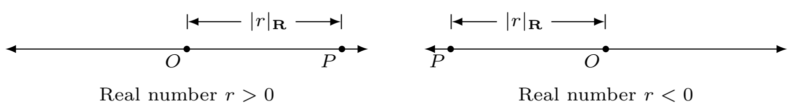
The main references are "Complex Analysis" by L. Ahlfors, "Complex Analysis" by J. Bak and D.J. Newman, Notes by J. Wetzel (University of Illinois), "Complex Analysis" by E.M. Stein and R. Shakarchi, and "Complex variables and applications" by J.W. Brown and R.V. Churchill

1 Facts about complex numbers

1.1 Beyond real numbers

Real numbers can be represented by points on a number line with a fixed origin O that represents the number 0. If r is a positive real number, we represent it as a point P to the right of O on the number line with distance r. If r is a negative real number, we represent it as a point to the left of O with distance $|r|_{\mathbf{R}}$, where

$$|r|_{\mathbf{R}} = \begin{cases} r & \text{if } r \geq 0 \\ -r & \text{if } r < 0. \end{cases}.$$



The set of real numbers is denoted by $\mathbf{R}$. Real numbers can be constructed rigorously using *Cauchy sequences* $\{s_n | n = 0, 1, 2 \cdots\}$ where s_n are rational numbers.

Let S be a subset of $\mathbf{R}$. We say that a number α is algebraic over S if α is a solution of a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where $a_j \in S, 0 \leq j \leq n$. For example, $1/2$ is a number algebraic over the set of integers $\mathbf{Z}$ since it satisfies $2x - 1 = 0$ and $\sqrt{2}$ is algebraic over $\mathbf{Q}$ and $\mathbf{Z}$ since it is a solution to $x^2 - 2 = 0$. Note that a number algebraic over S may or may not lie in S. For example, it is known that $\sqrt{2}$ is not rational even though it is algebraic over $\mathbf{Q}$.

Now, we know from the example of $\sqrt{2}$ that a number algebraic over $\mathbf{Q}$ is not rational. If we replace $\mathbf{Q}$ by $\mathbf{R}$, then we see that $\sqrt{2}$ is algebraic over $\mathbf{R}$ since it is a solution of $x - \sqrt{2} = 0$. The question that we want to ask is: Is there number algebraic over $\mathbf{R}$ which does not belong to $\mathbf{R}$? The answer to this question is yes. Suppose α is a solution of $x^2 + 1 = 0$ and α is real, then $\alpha^2 = -1 < 0$. But we know that this is impossible since $r^2 \geq 0$ for every $r \in \mathbf{R}$. The above discussion shows that there must be a set of numbers algebraic over $\mathbf{R}$ which properly contains $\mathbf{R}$. We will construct this set of numbers in the next section.

1.2 Complex numbers

In the previous section, we have seen that it is possible to construct “numbers” that are not in $\mathbf{R}$. We now define these “numbers” formally.

DEFINITION 1.1 Let $\mathbf{R}$ be the set of real numbers. The set of complex numbers $\mathbf{C}$ is the set of ordered pairs of real numbers (a, b) with addition and multiplication defined by

$$(a, b) + (c, d) = (a + c, b + d)$$

and

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

We also define a *scalar product* $\cdot$, namely, if $r \in \mathbf{R}$,

$$r \cdot (a, b) = (r \cdot a, r \cdot b).$$

Here $\cdot$ is the ordinary multiplication of real numbers.

With the definition of scalar product and $+$, we observe that the set $\mathbf{C}$ is a *vector space* of dimension 2 over $\mathbf{R}$.

We now consider $\cdot$. By definition of $\cdot$, we find that

$$(0, 1) \cdot (0, 1) = (-1, 0) = (-1) \cdot (1, 0).$$

With the notation $1 = (1, 0)$ and $i = (0, 1)$,

$$i \cdot i = (-1) \cdot 1.$$

We have thus found a “solution” to

$$x \cdot x = -1.$$

With the notation $1 = (1, 0)$ and $i = (0, 1)$, we can now write a complex number as

$$a \cdot 1 + b \cdot i.$$

Ignoring the colors for the operators, we arrive at the definition of complex numbers given in many textbooks.

The sum and product of complex numbers can now be written as

$$a + ib + c + id = (a + c) + i(b + d)$$

and

$$(a + ib)(c + id) = ac - bd + i(ad + bc)$$

respectively. Note that multiplication of complex numbers is motivated by treating $a + ib$ and $c + id$ like ordinary numbers with multiplication that distributes over addition.

DEFINITION 1.2 When a complex number is written as

$$z = a + ib,$$

we call a the *Real part* of z (denoted by $\operatorname{Re} z$) and b the *Imaginary part* of z (denoted by $\operatorname{Im} z$).

1.3 Division of complex numbers

When we first construct rational number, we ask for solution x such that

$$bx = a,$$

with $b \neq 0$. And we define $x = ab^{-1}$ and this leads to division of a by b. In a similar way, we ask for solution

$$(c + id)x = (a + ib).$$

We let

$$x = u + iv.$$

Then we must find u, v from comparing the real part and imaginary part of the numbers of both sides of the equation

$$cu - vd + i(cv + du) = a + ib.$$

The number u and v can be found using the relation

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

which gives

$$u + iv = \frac{ca + db}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.$$

Remark 1.1 From

$$cu - vd + i(cv + du) = a + ib,$$

we have

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} -v \\ u \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

This implies that we can identify $s + it$ with

$$\begin{pmatrix} s & -t \\ t & s \end{pmatrix}.$$

Remark 1.2 Division is usually done using "rationalization" by "multiplying" numerator and denominator by $c - id$ if we write

$$x = \frac{a + ib}{c + id}.$$

DEFINITION 1.3 Let z be a non-zero complex number. We define the multiplicative inverse of z, denoted by z^{-1}, as the complex number w satisfying the equation

$$wz = 1.$$

Note that by the division of complex numbers, we find that

$$(a + ib)^{-1} = \frac{1}{a^2 + b^2}(a - ib).$$

In mathematics, a group is a nonempty set G together with a binary operation $\circ : G \times G \to G$ that satisfies the following conditions:

1. There exists an element $e \in G$ such that $g \circ e = e \circ g = g$ for all $g \in G$,
2. For every $g \in G$, there exists $g' \in G$ such that $g \circ g' = g' \circ g = e$,
3. For all $g, h, k \in G$, $g \circ (h \circ k) = (g \circ h) \circ k$.

In the process of showing that $\mathbf{C}$ is a vector space of dimension 2 over $\mathbf{R}$, we would have shown that $(\mathbf{C}, +)$ is a group. With the multiplicative inverse defined, we can also show that $(\mathbf{C}\backslash\{0\}, \cdot)$ is a group provided that $\cdot$ is associative, which we leave as an exercise.

EXAMPLE 1.1 Show that if z, w and u are complex numbers then $z\cdot(u\cdot w) = (z\cdot u)\cdot w$ and $z\cdot u = u\cdot z$.

The addition and multiplication also satisfy the distributive law as can be verified in the following exercise.

EXAMPLE 1.2 Show that for complex numbers z, w and u,

$$z\cdot(w + u) = z\cdot w + z\cdot u.$$

Solution
Let $z = a + ib$, $w = c + id$ and $u = e + if$. Then left hand side is

$$(a + ib)\cdot(c + e + i(d + f)) = ac + ae - bd - bf + i(bc + be + ad + af).$$

The right hand side is

$$(a + ib)\cdot(c + id) + (a + ib)\cdot(e + if) = ac - bd + i(bc + ad) + ae - bf + i(be + af)$$

and both sides are the same.

The facts that $(\mathbf{C}, +)$ and $(\mathbf{C}\backslash\{0\}, \cdot)$ are abelian groups [1] and that $\cdot$ distributes over $+$ show that $(\mathbf{C}, +, \cdot)$ is a field.

1.4 Conjugate and modulus of z

There is a recurring appearance of the number $a^2 + b^2$ (in the inverse of z) and as determinant of

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

DEFINITION 1.4 The number

$$|z| = \sqrt{a^2 + b^2}$$

is called the modulus of z.

Note that $a^2 + b^2 = |z|^2$.

DEFINITION 1.5 The *conjugate* of z, denoted by $\overline{z}$, is defined by $a - ib$.

Note that

$$z\overline{z} = |z|^2 + i0. \tag{1.1}$$

In many textbooks, the above is written as

$$z\overline{z} = |z|^2. \tag{1.2}$$

This is not accurate since by (1.1), we know that

$$|z|^2 = \mathrm{Re}(z\overline{z}).$$

However, once we are familiar with complex numbers, we will not distinguish $z\cdot\overline{z}$ from $|z|^2$ and use (1.2) instead.

[1] Abelian groups are groups with binary operation having the additional property that $g \circ g' = g' \circ g$.

Remark 1.3 Note that $|\cdot|$ is consistent with absolute value $|\cdot|_{\mathbf{R}}$, which we encountered in the case of real numbers. Note that

$$|a + i \cdot 0| = \sqrt{a^2} = |a|_{\mathbf{R}}.$$

EXAMPLE 1.3 Establish the following facts:

$$\operatorname{Re} z = \frac{1}{2}(z + \overline{z}), \operatorname{Im} z = \frac{1}{2i}(z - \overline{z}), \overline{(z+w)} = \overline{z} + \overline{w},$$
$$\overline{zw} = \overline{z} \cdot \overline{w}, |zw| = |z||w|, |z/w| = |z|/|w| \quad \text{if } w \neq 0, |\overline{z}| = |z|.$$

EXAMPLE 1.4 Show that if $z, w \in \mathbf{C}$, then $\overline{zw} = \overline{z} \cdot \overline{w}$ and $|zw| = |z||w|$.

Solution

The first identity can be proved directly. Let $z = a + ib$ and $w = c + id$. Then

$$\overline{z \cdot w} = \overline{ac - bd + i(bc + ad)} = ac - bd - i(bc + ad).$$

On the other hand,

$$\overline{z} \cdot \overline{w} = (a - ib)(c - id) = ac - bd - i(bc + ad).$$

Note that

$$|zw|^2 = (zw)\overline{zw} = zw\overline{zw}$$
$$= z\overline{z}w\overline{w} = |z|^2 \cdot |w|^2,$$

which concludes the proof of the second identity.

EXAMPLE 1.5 Show that if $w = 0$ if and only if $|w| = 0$ and deduce that $zw = 0$ implies that $z = 0$ or $w = 0$.

Solution

If $w = 0$, then $|w| = 0^2 + 0^2 = 0$. If $|w| = 0$ and $w = a + ib$, then $0 = |w|^2 = a^2 + b^2 \geq a^2 \geq 0$. This implies that $a = 0$. Similarly, $b = 0$. Now if $zw = 0$, then $|zw| = |z||w| = 0$. This implies that $|z| = 0$ or $|w| = 0$. By previous observation, we conclude that $z = 0$ or $w = 0$. Another way to solve $zw = 0$ implies $z = 0$ or $w = 0$ is to multiply both sides of $zw = 0$ by the inverse of z if $z \neq 0$.

EXAMPLE 1.6 Show that if A and B are integers that can be written as a sum of two squares then AB is a sum of two squares.

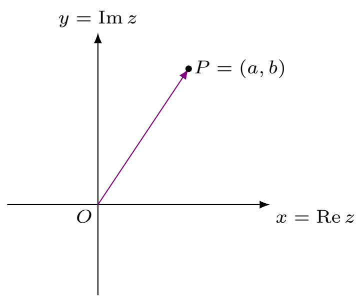
Solution

Let $m = a^2 + b^2 = |a + ib|^2$ and $n = c^2 + d^2 = |c + id|^2$. Then

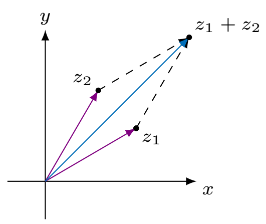
$$mn = |a + ib|^2 \cdot |c + id|^2 = |(a + ib)(c + id)|^2 = (ac - bd)^2 + (ad + bc)^2.$$

1.5 The complex plane

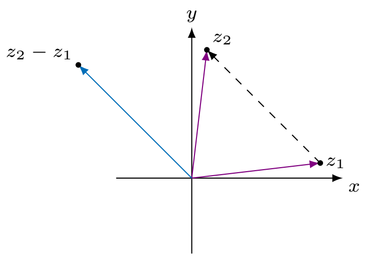
As seen in previous sections, a complex number is defined as a number of the form (a, b). Just as a real number can be represented graphically by a point on a number line, a complex number can be represented by a point on a plane. This is illustrated in the following diagram:



Addition of complex numbers then corresponds to addition of “vectors” as shown in the following diagram:



The following diagram illustrates the difference of two complex numbers:



It is known that if $|r|_{\mathbf{R}}$ is the absolute value of a real number r then

$$|r+s|_{\mathbf{R}} \leq |r|_{\mathbf{R}} + |s|_{\mathbf{R}}.$$

The same is true for the modulus of complex numbers. More precisely, we have the following triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{1.3}$$

To see (1.3) geometrically, we first observe that the modulus $|z|$ is the length of the vector (a, b) that represents z. Now, note that z_1, z_2 and $z_1 + z_2$ form the vertices of a triangle. Since $|z_1 + z_2|$ is the length of a side of the triangle, it must be less than or equal to the sum of the lengths of the other two sides and this gives (1.3). We now give an algebraic proof of (1.3).

Proof of (1.3)
First, we observe that

$$\operatorname{Re} z \leq |z|.$$

This follows from the fact that if $z = x + iy$, then

$$x \leq |x| \leq \sqrt{x^2 + y^2}. \tag{1.4}$$

Next, observe that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2}).$$

Hence,

$$|z_1 + z_2|^2 = z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2}.$$

But by (1.4),

$$z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\mathrm{Re}(z_1\overline{z_2}) \leq 2|z_1\overline{z_2}| = 2|z_1| \cdot |z_2|.$$

Therefore,

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

□

COROLLARY 1.1 Let z_1 and z_2 be complex numbers. Then

$$||z_1| - |z_2|| \leq |z_1 - z_2|.$$

Proof
We have $|z + w| \leq |z| + |w|$. Take $z = z_2, w = z_1 - z_2$. Then

$$|z_1| \leq |z_2| + |z_1 - z_2|.$$

Interchanging z_1 and z_2, we find that

$$|z_1| - |z_2| \geq -|z_2 - z_1|$$

and this completes the proof. □

Note that if we treat z_1 and z_2 as points on the complex plane, then the "vector" formed by z_1 and z_2 beginning with z_1 has "position vector" determined by $z_2 - z_1$. This vector has length $|z_2 - z_1|$ and this is the distance between z_1 and z_2.

EXAMPLE 1.7 Show that if $|z| = 2$ then

$$|\text{Im}\,(1 - \overline{z} + z^2)| \leq 6.$$

Solution
Note that since 1 is real,

$$\text{Im}\,(1 - \overline{z} + z^2) = \text{Im}\,(-\overline{z} + z^2).$$

We have shown in the proof of the triangle inequality that $|\text{Re}\, z| \leq |z|$. The same is true when we replace Re by Im. Hence we conclude that

$$|\text{Im}\,(1 - \overline{z} + z^2)| = |\text{Im}\,(-\overline{z} + z^2)| \leq |-\overline{z} + z^2| \leq |\overline{z}| + |z|^2 = 6.$$

EXAMPLE 1.8 Let $z, w, u, v \in \mathbf{C}$ with $|u| \neq |v|$. Show that

$$\frac{\text{Re}(z + w)}{|u + v|} \leq \frac{|z| + |w|}{||u| - |v||}.$$

Solution
Since

$$\text{Re}(z + w) \leq |z + w| \leq |z| + |w|$$

and

$$||u| - |v|| = ||u| - |-v|| \leq |u - (-v)| = |u + v|,$$

the inequality follows.

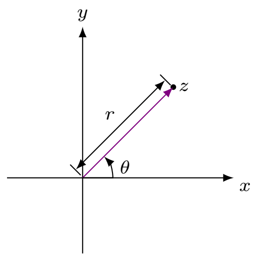
1.6 The polar representation of complex numbers

Given a point in the plane which is not the origin, we may represent the point using polar coordinate system rather than the usual rectangular coordinate system. Let (x, y) be a point in the first quadrant. Then the complex number $z = x + iy$ can be written as

$$z = r(\cos\theta + i\sin\theta).$$

We will write

$$\operatorname{cis}\theta = \cos\theta + i\sin\theta.$$



When z is not in the first quadrant, we use the more general definitions of sine and cosine for obtuse angles and negative angles. This allows us to write in general,

$$z = r\operatorname{cis}\theta.$$

Note that θ is not unique since $\sin x$ and $\cos x$ are both periodic with period 2π.

Given z, we use **arg** z to denote the set $\{\theta | z = r\operatorname{cis}\theta\}$. This set is an equivalence class obtained from the equivalence relation $u \sim v$ if and only if $u = v + 2k\pi$ for some integer k. The principal value of **arg** z, denoted by $\operatorname{Arg} z$, is the representative of **arg** z that lies in $(-\pi, \pi]$.

EXAMPLE 1.9 Let $z = i$. Determine $|z|$, **arg** z, Arg z and express z in polar coordinates.

Solution
The answers are $|z| = 1$, $\mathbf{arg}\, z = \{\pi/2 + 2k\pi, k \in \mathbf{Z}\}$, $\operatorname{Arg} z = \pi/2$, and $i = \operatorname{cis}\dfrac{\pi}{2}$.

EXAMPLE 1.10 Sketch the following regions representing the following sets in a complex plane:

(a) $\{z | \operatorname{Re} z > 0\}$
(b) $\{z | -\pi/3 < \operatorname{Arg} z < \pi/3\}$
(c) $\{z || z+1| < 1\}$

THEOREM 1.2 Let $z = r\operatorname{cis}\theta$ and $w = s\operatorname{cis}\psi$. Then

$$zw = rs\operatorname{cis}(\theta + \psi).$$

that is,

$$|zw| = |z||w|$$

and

$$\mathbf{arg}\, zw = \mathbf{arg}\, z + \mathbf{arg}\, w. \tag{1.5}$$

Here, the addition of $\mathbf{arg}\, z$ and $\mathbf{arg}\, w$ is defined as $\{\theta + \psi | zw = rs\text{cis}(\theta + \psi)\}$ and (1.5) means that

$$\text{Arg}(zw) = \text{Arg}\, z + \text{Arg}\, w + 2k\pi$$

for some integer k.

Proof
We have

$$\begin{aligned} zw &= rs(\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi) \\ &= rs(\cos\theta\cos\psi - \sin\theta\sin\psi + i(\cos\theta\sin\psi + \sin\theta\cos\psi)) \\ &= rs(\cos(\theta+\psi) + i\sin(\theta+\psi)), \end{aligned}$$

where we have used the standard sine and cosine formula for addition of angles, namely,

$$\cos(\theta + \psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$$

and

$$\sin(\theta + \psi) = \cos\theta\sin\psi + \sin\theta\cos\psi.$$

□

Using induction, we deduce that

COROLLARY 1.3 Let $z_1, z_2, \cdots, z_n \in \mathbf{C}$. Then

$$|z_1 z_2 \cdots z_n| = |z_1| \cdot |z_2| \cdots |z_n|,$$

and

$$\mathbf{arg}\,(z_1 z_2 \cdots z_n) = \mathbf{arg}\, z_1 + \cdots + \mathbf{arg}\, z_n.$$

In Corollary 1.3, we let $z_1 = z_2 = \cdots = z$ and conclude that

COROLLARY 1.4 Let n be an integer. Then

$$(\text{cis}\,\theta)^n = \text{cis}\, n\theta.$$

Remark 1.4 The relation in Theorem 1.4 holds only for integers. It is not true for rational numbers as one would encounter the notion of multi-valued function such as $(\cos x + i \sin x)^{1/n}$, $n \in \mathbf{Z}$.

EXAMPLE 1.11 Find all complex numbers z satisfying $z^3 = 1$.

Solution
Write $1 = \cos\theta + i\sin\theta$ where $\theta = 2k\pi, k \in \mathbf{Z}$. Write $z = r(\cos\psi + i\sin\psi)$. By Corollary 1.4,

$$z^3 = r^3 (\cos 3\psi + i \sin 3\psi) .$$

Since $z^3 = 1$, we conclude that $r = 1$ and $3\psi = 2k\pi, k \in \mathbf{Z}$. Therefore, $\psi = 2k\pi/3, k \in \mathbf{Z}$. This gives rise to three distinct solutions of ψ in $(-\pi, \pi]$ and they are $-2\pi/3, 0$, and $2\pi/3$.

EXAMPLE 1.12 Let $0 < \theta < \pi$. Derive the Lagrange's trigonometric identity

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2\sin(\theta/2)}.$$

Solution
The left hand side is

$$\mathrm{Re}\,(1 + z + z^2 + \cdots + z^n) = \mathrm{Re}\left(\frac{z^{n+1}-1}{z-1}\right),$$

where $z = \cos\theta + i\sin\theta$. Now, observe that

$$\begin{aligned}\frac{z^{n+1}-1}{z-1} &= \frac{(z^{n+1}-1)(\overline{z}-1)}{|z-1|^2} \\ &= \frac{z^n - \overline{z} + 1 - z^{n+1}}{2(1-\cos\theta)},\end{aligned}$$

where we have used the fact that $z\overline{z} = 1$. Hence,

$$\begin{aligned}\mathrm{Re}\left(\frac{z^{n+1}-1}{z-1}\right) &= \mathrm{Re}\left(\frac{z^n - \overline{z} + 1 - z^{n+1}}{2(1-\cos\theta)}\right) \\ &= \frac{\cos n\theta - \cos\theta + 1 - \cos(n+1)\theta}{2(1-\cos\theta)} \\ &= \frac{1}{2} + \frac{1}{4\sin^2(\theta/2)}\left(\cos(n\theta) - \cos((n+1)\theta)\right).\end{aligned}$$

The identity we want to prove now follows using

$$\cos(n\theta) - \cos((n+1)\theta) = -2\sin((2n+1)\theta/2)\sin(-\theta/2).$$

Remark 1.5 There is another way of proving the above identity. One needs only to observe that for positive integer ℓ,

$$\sin(\theta/2)\cos\ell\theta = \frac{1}{2}\left(\sin((\ell+1/2)\theta) - \sin((\ell-1/2)\theta)\right)$$

and that

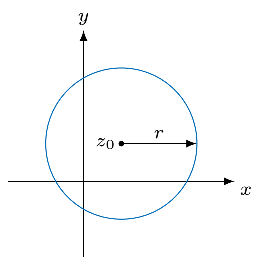
$$\sum_{\ell=1}^{n}\sin(\theta/2)\cos\ell\theta = \frac{1}{2}\left(\sin((n+1/2)\theta) - \sin(\theta/2)\right).$$

1.7 Sets in the complex plane

When we study the real numbers, we define objects such as open intervals, closed intervals and bounded sets. In this section, we give definitions to analogous objects for the complex numbers. Most of these definitions look intimidating at first sight but they define very natural objects.

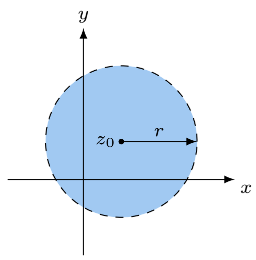
1. A circle of radius $r > 0$ centered at z_0 is the set

$$C(z_0; r) := \{z \mid |z - z_0| = r\}.$$



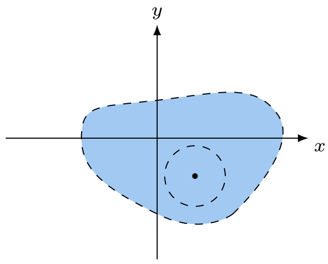
2. A ball of radius r and center z_0 is the set

$$B(z_0; r) := \{z \mid |z - z_0| < r\}.$$



The dotted line is used to indicate that the circle $|z - z_0| = r$ is not in the set $B(z_0; r)$.

3. A subset S of $\mathbf{C}$ is said to be *open* in $\mathbf{C}$ if for any $z \in S$ there exist a $\delta > 0$ such that $B(z;\delta) \subset S$. We also say that S is an *open set.*



A ball of radius r with center z_0 is an open set.
The set

$$S = \{z| -\frac{\pi}{4} < \operatorname{Arg} z < \frac{\pi}{4}\}$$

is open. In general, one may visualize open sets in $\mathbf{C}$ as shaded regions with dotted boundaries.

There are sets which are not open. Examples of such sets are $C(z_0;r)$ and $\{z|\operatorname{Re} z \geq 1\}$.

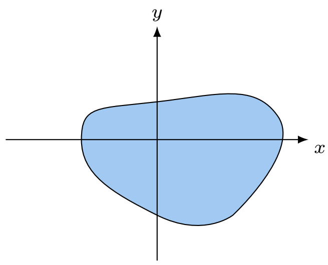
4. For any set S, let

$$S^c = \mathbf{C}\backslash S = \mathbf{C} - S$$

be the complement of S in $\mathbf{C}$, that is,

$$S^c = \{z \in \mathbf{C}| z \notin S\}.$$

A subset S of $\mathbf{C}$ is said to be *closed* if the complement of S in $\mathbf{C}$, denoted by S^c, is open. Examples of closed sets are $C(z_0;r)$ and $\{z|\operatorname{Re} z \geq 1\}$. The following illustrates an example of a closed set:



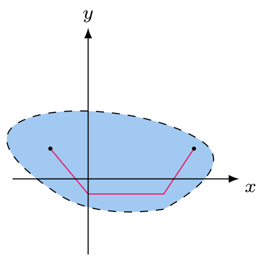
Remark 1.6 A set S that is not open is not necessarily closed. The set $\{0 \leq \operatorname{Re} z < 1\}$ is one such example.

5. Let S be a subset of $\mathbf{C}$. The set of points B with the property that every open ball of the form $B(z_0;r)$, with $z_0 \in B$, has non-empty intersection with S and S^c. The set B is called the *boundary* of S and the notation for this set is ∂S.

6. A set S is *bounded* if it is contained in a ball $B(0; M)$ for some $M > 0$.
7. Let $[z_1, z_2]$ denotes the line segment with endpoints z_1 and z_2. A polygonal line is a finite union of line segments of the form

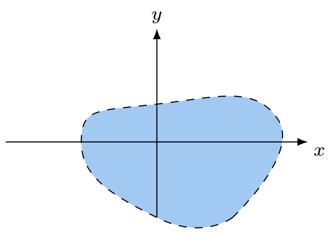
$$[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n].$$

   If any two points of S can be connected by a polygonal line contained in S, S is said to be *polygonally connected*. The following set is an example of a polygonally connected set.

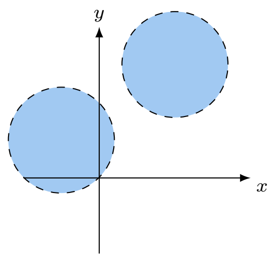


8. A nonempty *open polygonally connected* set in $\mathbf{C}$ will be called a *region*.

   In this course, we will study functions defined on a region.
9. A set S in $\mathbf{C}$ is said to be *disconnected* if there exist two disjoint open sets A and B in $\mathbf{C}$ such that $S = (S \cap A) \cup (S \cap B)$ and that neither A nor B above contains S. If S is not disconnected it is said to be *connected* (see the following diagrams).



Connected set



Disconnected set

Remark 1.7 It can be shown that in $\mathbf{C}$, *an open set is polygonally connected if and only if it is connected.* For more details see [p. 54, Ahlfors]. The usual definition of a region is a non-empty open connected set.

1.8 Appendix: Set of complex numbers as topological space

I hope students will read this section. It is my attempt to introduce topological spaces.

DEFINITION 1.6 A *topological space* $T = \{A, \mathcal{T}\}$ consists of a non-empty set A together with a fixed collection $\mathcal{T}$ of subsets satisfying

(T1) $A, \phi \in \mathcal{T}$
(T2) the intersection of any two sets in $\mathcal{T}$ is again in $\mathcal{T}$
(T3) the union of any collection of sets in $\mathcal{T}$ is again in $\mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* for A and the members of $\mathcal{T}$ are called the open sets of T.

EXAMPLE 1.13 Let $A = \{a, b\}$ and $\mathcal{T} = \{A, \phi, \{a\}\}$. Then $(A, \mathcal{T})$ is a topological space.

As one can see from the example, topological space can be very strange.

In the case of complex numbers, we declare set U to be "open in $\mathbf{C}$" if for each $z \in U$, then there exists an $\epsilon > 0$ such that $B(z; \epsilon) \subset U$. (Note that we are NOT saying that all examples of open sets must arise from this definition. See the above example.)

We let $\mathcal{T}$ to be the collection of "open sets U in $\mathbf{C}$". We now check that $(\mathbf{C}, \mathcal{T})$ is a topological space. Note that (T1) is clearly satisfied.

For (T2), let U_1 and U_2 be open set in $\mathbf{C}$. Let $z \in U_1 \cap U_2$. Then $z \in U_1$ and $z \in U_2$. Since U_1 is open in $\mathbf{C}$, there exists an $\epsilon_1 > 0$ such that $B(z; \epsilon_1) \subset U_1$. Similarly, since U_2 is open in $\mathbf{C}$, there exists an $\epsilon_2 > 0$ such that $B(z; \epsilon_2) \subset U_2$. Choose $\epsilon = \min(\epsilon_1, \epsilon_2)$. Then $B(z; \epsilon) \subset U_1 \cap U_2$ and therefore, $U_1 \cap U_2$ is open.

For condition (T3), if z is in a union of open sets, say A, where

$$A = \bigcup_i U_i,$$

then $z \in U_1$, say. Since U_1 is open in $\mathbf{C}$, there exists an $\epsilon > 0$ such that

$$B(z; \epsilon) \subset U_1 \subset A.$$

Hence, A is open and

$$\bigcup_i U_i \in \mathcal{T}.$$

In conclusion, we have $(\mathbf{C}, \mathcal{T})$ is a topological space.

2 Analytic Functions

2.1 Functions of a complex variable

When we first encounter real numbers, we define a function on $S \subset \mathbf{R}$ as a rule that assigns to each $x \in S$, a unique real number $y \in \mathbf{R}$. For example the rule $f(x) = x^2$ sends a real number to its square.

In a similar manner, we define a function on a set $S \subset \mathbf{C}$ to be a rule that assigns $z \in S$, a complex number $w \in \mathbf{C}$. The number w is called the value of f at z or image of z under f and is denoted by

$$w = f(z).$$

The set S is called the domain of definition of f. A function f is sometimes referred to as a single-valued function because f sends each z in its domain of definition to exactly one number. There are rules which assign a complex number z to more than one number. We referred to such a rule as a multi-valued function. Examples of multi-valued functions are $z^{1/2}$ and $z^{1/3}$, where $z^{1/2}$ and $z^{1/3}$ are complex numbers u satisfying

$$u^2 = z \quad \text{and} \quad u^3 = z,$$

respectively. The rule $\arg(z)$ that assigns z to the set of angles θ (viewed as a real number embedded in $\mathbf{C}$) satisfying $z = r\text{cis}(\theta)$ is a multi-valued function. In this course, we concentrate on rules that assign z to a unique complex number $w = f(z)$. In other words, we assume that our rules are functions or more precisely, single-valued functions. We will return to multi-valued functions when we discuss the "complex version" of log.

To define a function, both the domain of definition and the rule must be given. The function

$$f(z) = \frac{1}{z}$$

is defined on $\mathbf{C}\backslash\{0\}$ since $f(z)$ is not defined at 0.

EXAMPLE 2.1 Let $f(z) = z^2$. Note that in terms of x and y coordinate,

$$f(z) = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Suppose $w = u + iv$ is the value of a function of $f(z)$ at $z = x + iy$. Then u and v depend on the real variables x and y and we find that

$$f(z) = w = u(x, y) + iv(x, y).$$

EXAMPLE 2.2 When $f(z) = z^2$,

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

This implies that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

EXAMPLE 2.3 Let $f(z) = z^3$. Then

$$f(x + iy) = (x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3).$$

This implies that

$$u(x, y) = x^3 - 3xy^2 \quad \text{and} \quad v(x, y) = 3x^2y - y^3.$$

EXAMPLE 2.4 Let $f(z) = \dfrac{1}{1 - |z|^2} = \dfrac{1}{1 - x^2 - y^2}$. Then

$$u(x, y) = \frac{1}{1 - x^2 - y^2} \quad \text{and} \quad v(x, y) = 0.$$

EXAMPLE 2.5 Let

$$f(z) = \frac{z}{z + \bar{z}} = \frac{x + iy}{2x}.$$

Then

$$u(x, y) = \frac{1}{2} \quad \text{and} \quad v(x, y) = \frac{y}{2x}.$$

2.2 Limits

In the case of a real variable, we say that

$$\lim_{x \to x_0} f(x) = w_0$$

if for every $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that

$$|f(x) - w_0|_{\mathbf{R}} < \epsilon \quad \text{whenever} \quad 0 < |x - x_0|_{\mathbf{R}} < \delta_\epsilon.$$

In the case of complex variable, the definition is similar.

DEFINITION 2.1 We say that

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if for every $\epsilon > 0$, there exist a $\delta_\epsilon > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_\epsilon.$$

Sometimes, we write

$$(z) \to w_0 \quad \text{for} \quad z \to z_0$$

to represent

$$\lim_{z \to z_0} f(z) = w_0.$$

If w_0 is ∞, we write $f(z) \to \infty$ for $z \to z_0$ if for every $M > 0$, there exist $\delta_M > 0$ such that

$$|f(z)| > M \quad \text{whenever} \quad 0 < |z - z_0| < \delta_M.$$

We also write $f(z) \to \infty$ for $z \to \infty$ if for every $M > -$, there exist $K_M > 0$ such that

$$|f(z)| > M \text{ whenever } |z| > K_M.$$

EXAMPLE 2.6 Let $f(z) = \dfrac{iz}{2}$. Show that

$$\lim_{z \to 1} f(z) = \frac{i}{2},$$

using the definition of limit.

To show that the above holds, we need to show that for each $\epsilon > 0$, there exist a $\delta_\epsilon > 0$ such that

$$\left|\frac{iz}{2} - \frac{i}{2}\right| < \epsilon \quad \text{whenever} \quad 0 < |z-1| < \delta_\epsilon.$$

We usually work "backwards" to obtain our δ_ϵ. Let $\epsilon > 0$ be arbitrarily chosen. Now,

$$\left|\frac{iz}{2} - \frac{i}{2}\right| < \epsilon$$

if and only if

$$\left|\frac{i}{2}\right| |z-1| < \epsilon$$

if and only if

$$0 < |z-1| < 2\epsilon$$

since $|i| = 1$. Hence, we may take our δ_ϵ to be 2ϵ. Therefore, $f(z) \to \dfrac{i}{2}$ when $z \to 1$.

EXAMPLE 2.7 Show that the limit of the function $f(z) = \dfrac{\overline{z}}{z}$ as $z \to 0$ does not exist.

Solution

Let $z = 0 + it$ and we see that $\overline{z}/z = -it/it$ tends to -1 as z tends to 0 along the imaginary axis. Let $z = t$ and $\overline{z}/z$ tends to 1 as z tends to 0 along the real axis. Hence, the limit of $\overline{z}/z$ does not exist as z tends to 0.

EXAMPLE 2.8 Let $f(z) = 1/z$. Then

$$\lim_{z\to 0} f(z) = \infty.$$

We say that $f(z)$ has a pole at $z = 0$.

EXAMPLE 2.9 Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0$$

with $a_j \in \mathbf{C}$. Show that $P(z) \to \infty$ as $z \to \infty$.

Solution

Let $n \geq 1$ be an integer and

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n,$$

with $a_n \neq 0$. If $n = 1$, then it is clear that

$$|a_1 z + a_0| > (|a_1||z| - |a_0|) \to \infty \quad \text{if} \quad |z| \to \infty.$$

Now, let $n > 1$. Let $A = \max_{0 \leq i \leq n-1}(|a_i|)$. Then

$$|P(z)| \geq |a_n z^n| - |a_{n-1} z^{n-1} - \cdots - |a_0| \geq |a_n z^n| - A\left(|z|^{n-1} + \cdots + |z| + 1\right).$$

Since $|z| \to \infty$, we may assume $|z| > 1$. This implies that

$$|P(z)| \geq |a_n z^n| - nA|z|^{n-1} = |z|^{n-1}\left(|a_n z| - nA\right) > M,$$

whenever

$$|z| > \max\left(1, \frac{nA+1}{|a_n|}, M^{1/(n-1)}\right).$$

2.3 Theorems on Limits

Most of the results in this section are familiar as they also appear in the same form in Calculus and Real Analysis.

If

$$\lim_{z \to z_0} f(z) = A \quad \text{and} \quad \lim_{z \to z_0} g(z) = B,$$

then

$$\lim_{z \to z_0} (f(z) + g(z)) = A + B. \tag{2.1}$$

The proof of this is similar to that for real variable.

Suppose $f(z) \to w_0$ as $z \to z_0$, then

$$\lim_{z \to z_0} \operatorname{Re} f(z) = u_0$$

and

$$\lim_{z \to z_0} \operatorname{Im} f(z) = v_0,$$

where $w_0 = u_0 + iv_0$. This is because for $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_\epsilon.$$

Now, observe that

$$|f(z) - w_0| = |\overline{f(z)} - \overline{w_0}|$$

and hence, we find that $\overline{f(z)} \to \overline{w_0}$. Now,

$$\mathrm{Re}(f(z)) = \frac{f(z) + \overline{f(z)}}{2}.$$

Therefore, by (2.1)

$$\lim_{z\to z_0} \mathrm{Re}(f(z)) = \frac{w_0 + \overline{w_0}}{2} = \mathrm{Re}(w_0) = u_0.$$

Similarly,

$$\lim_{z\to z_0} \mathrm{Im}(f(z)) = \mathrm{Im}(w_0) = v_0.$$

Conversely, if $f(z) = u(x, y) + iv(x, y)$ and $F(z) := u(x, y) \to u_0$ and $G(z) = v(x, y) \to v_0$ (Note that $2x = z + \overline{z}$ and $2iy = z - \overline{z}$ and so, $u(x, y)$ and $v(x, y)$ are functions of z.) as $z \to z_0$, then by (2.1), we deduce that $f(z) = F(z) + iG(z) \to u_0 + iv_0$ as $z \to z_0$. We have thus proved the following theorem :

THEOREM 2.1 Suppose $f(z) = u(x, y) + iv(x, y), z_0 = x_0 + iy_0, w_0 = u_0 + iv_0$. Then

$$\lim_{z\to z_0} f(z) = w_0,$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

THEOREM 2.2 Suppose $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$. Then

(a) $\lim_{z\to z_0} f(z)g(z) = AB$;

(b) If $B \neq 0$, then $\lim_{z\to z_0} f(z)/g(z) = A/B$.

2.4 Continuity

DEFINITION 2.2 A function $f(z)$ is said to be continuous at z_0 if

(a) $\lim_{z\to z_0} f(z)$ exists,

(b) $f(z_0)$ exists, and

(c) $\lim_{z\to z_0} f(z) = f(z_0)$.

Statement (c) says that for every $\epsilon > 0$ there exist a δ_ϵ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta_\epsilon.$$

In other words, $f(z)$ is continuous at z_0 if we obtain the limit $\lim_{z\to z_0} f(z)$ by simply substituting z_0 into $f(z)$.

DEFINITION 2.3 A function $f(z)$ is said to be continuous in a domain D if it is continuous at every point in D.

Theorem 2.1 says that $f(z) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$ if and only if the corresponding real and imaginary parts are continuous at (x_0, y_0), i.e.,

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u(x_0, y_0), \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v(x_0, y_0).$$

Hence we may decide if a complex valued function $f(z)$ is continuous at any point z_0 by using known results in Calculus of two variables.

EXAMPLE 2.10 The function $f(z) = xy^2 + i(2x - y)$ is continuous at every point (x, y) since the corresponding real and imaginary parts are continuous functions of two variables.

As in the case of real variables, we have the fact that if $f(z)$ and $g(z)$ are continuous, then the functions $f(z) + g(z)$, $f(z)g(z)$, $f(z)/g(z)$ and $f(g(z))$ are continuous in the domain for which the functions are defined.

EXAMPLE 2.11 The polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is continuous for all $z \in \mathbf{C}$.

Solution

The function z is continuous implies that z^2 is continuous since product of continuous functions is continuous. By induction, z^k is continuous for all positive integers k. Now, sum of continuous functions is continuous, hence every polynomial in z is continuous.

EXAMPLE 2.12 Show that if $f(z)$ is continuous at z_0 then $|f(z)|$ is continuous at z_0.

Solution
By triangle inequality,

$$|f(z)| - |f(z_0)| < |f(z) - f(z_0)|$$

and

$$|f(z_0)| - |f(z)| < |f(z) - f(z_0)|.$$

Hence,

$$||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)|.$$

The result follows since

$$|f(z) - f(z_0)| < \epsilon$$

implies that

$$||f(z)| - |f(z_0)|| < \epsilon.$$

Remark 2.1 Strictly speaking, $|f|$ is a function from **C** to **R**. However, we may identify this function with $F = |f| + i \cdot 0$ and conclude that F is continuous on **C**.

Remark 2.2 The notion of continuity can be generalized to arbitrary topological spaces other than **C**. In order to achieve that, one has to define continuous function based on open sets instead of open balls. For more details, see the Appendix.

2.5 Derivative

The derivative of a function f at a (in the real variable case), denoted by $f'(a)$, is defined as the limit (if it exists)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

For the case of complex variable, the definition is similar :

DEFINITION 2.4 The derivative of a function f at z, denoted by $f'(z)$, is defined as the limit (if it exists)

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}.$$

We may also write

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

EXAMPLE 2.13 Show that the function

$$f(z) = \overline{z}$$

is not differentiable at any point z.

Solution
Observe that

$$\frac{\overline{z + h} - \overline{z}}{h} = \frac{\overline{h}}{h}.$$

By Example 2.7, we find that

$$\lim_{h \to 0} \frac{\overline{h}}{h}$$

does not exist and so $f(z)$ is not differentiable at any point z.

EXAMPLE 2.14 For a given $z_0 \in \mathbf{C}$, discuss the differentiability of $f(z) = |z|^2$ at $z = z_0$.

Solution
Observe that

$$\begin{aligned} \frac{|z + h|^2 - |z|^2}{h} &= \frac{(z + h)\overline{(z + h)} - z\overline{z}}{h} \\ &= \frac{h\overline{z} + \overline{h}z + h\overline{h}}{h} \\ &= \overline{z} + z\frac{\overline{h}}{h} + \overline{h}. \end{aligned}$$

If $z = 0$, then the limit exists and $f'(0) = 0$. If $z \neq 0$ and $h = \sigma \in \mathbf{R}$, then we find that

$$\overline{z} + z\frac{\overline{\sigma}}{\sigma} + \overline{\sigma} \to \overline{z} + z \quad \text{as } \sigma \to 0.$$

Similarly, if $h = it$ with $t \in \mathbf{R}$, then

$$\overline{z} + z\frac{\overline{it}}{it} + \overline{it} \to \overline{z} - z \quad \text{as } t \to 0.$$

In other words, if $z \neq 0$, the limit $\lim\limits_{h\to 0} \dfrac{|z+h|^2 - |z|^2}{h}$ does not exist. In conclusion, the derivative of $f(z)$ is defined only at $z = 0$.

Note that both functions $\overline{z}$ and $|z|^2$ are continuous functions on $\mathbf{C}$. But $f(z) = \overline{z}$ is not differentiable everywhere while $f(z) = |z|^2$ is differentiable only at $z = 0$. It turns out that as in the case of real variable, differentiable functions are continuous. The proof is exactly the same as that for the real variable case.

The rules for differentiating a complex valued functions are the same as those for real variables. This is because the definitions for $f'(x)$ and $f'(z)$ are similar. (For example, to prove that $f'(z) = 2z$ when $f(z) = z^2$, the proof is similar to functions on real numbers.) As such, we leave it as an exercise for the readers to prove the following standard results for differentiation:

1. $\dfrac{d}{dz}c = 0$
2. $\dfrac{d}{dz}z^n = nz^{n-1}, n \in \mathbf{Z}^+$
3. $\dfrac{d}{dz}(f(z) + F(z)) = f'(z) + F'(z)$
4. $\dfrac{d}{dz}(f(z)F(z)) = f(z)F'(z) + f'(z)F(z)$
5. (Chain Rule) If $F(z) = g(f(z))$, then $\dfrac{d}{dz}F(z) = g'(f(z))f'(z)$.

2.6 Cauchy-Riemann equations

We first recall the definition of partial derivatives. If $F(x, y)$ is a function of two real variables then

$$\frac{\partial F}{\partial x} = \lim_{\delta x \to 0} \frac{F(x + \delta x, y) - F(x, y)}{\delta x}$$

whenever the limit exists. Similarly,

$$\frac{\partial F}{\partial y} = \lim_{\delta y \to 0} \frac{F(x, y + \delta y) - F(x, y)}{\delta y}.$$

We recall that

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

If the above limit exists, then regardless of how h approaches 0, the resulting value would be the same. We now let h approaches 0 along the real axis. Write $h = \delta x$. Write $f = u(x, y) + iv(x, y)$. Let $h = \delta x$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x + \delta x, y) + iv(x + \delta x, y) - u(x, y) - iv(x, y)}{\delta x}.$$

Therefore,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Next, we let h approaches 0 along the imaginary axis. Write $h = i\delta y$,, we have

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x, y + \delta y) + iv(x, y + \delta y) - u(x, y) - iv(x, y)}{i\delta y}.$$

Therefore,

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Hence,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

and

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

We conclude that the Cauchy-Riemann equations hold, namely,

$$u_x = v_y \quad \text{and} \quad v_x = -u_y,$$

where $F_t = \dfrac{\partial F}{\partial t}$.

EXAMPLE 2.15 Verify that the function $f(z) = z^4$ satisfies the Cauchy-Riemann equations. (This is not surprising since all polynomials in z are differentiable and hence, Cauchy-Riemann equations are satisfied.)

Solution

In this case, $u(x, y) = x^4 - 6x^2y^2 + y^4$ and $v = 4x^3y - 4y^3x$. One checks easily that u and v satisfy the Cauchy-Riemann equations.

Remark 2.3 Note that we can create infinitely many functions u and v satisfying the Cauchy-Riemann equations by considering functions of the form z^n, where n is any positive integer.

EXAMPLE 2.16 Suppose $f(z) = e^{-y}\sin x + iv(x, y)$ is differentiable everywhere. Find $v(x, y)$.

Solution

Let $u = e^{-y}\sin x$. Then $u_x = e^{-y}\cos x$. Since the function is differentiable everywhere, the Cauchy-Riemann equations must be satisfied. Hence,

$$v_y = u_x = e^{-y}\cos x.$$

We conclude that

$$v = -e^{-y}\cos x + C(x),$$

where $C(x)$ is a function of x.

Next, $u_y = -e^{-y}\sin x$. Hence

$$v_x = e^{-y}\sin x.$$

Therefore,

$$v = -e^{-y}\cos x + D(y),$$

where $D(y)$ is a function of y. But $C(x) = D(y)$ implies that $C(0) = D(y)$ for all $y \in \mathbf{R}$ and so, $D(y)$ is a constant. Therefore, $C(x) = D(y) = a$ with $a \in \mathbf{C}$. We therefore conclude that

$$f(z) = e^{-y}\sin x - ie^{-y}\cos x + ia.$$

We have seen that if $f = u + iv$ and f is differentiable at $z = z_0$, then u and v satisfies the Cauchy-Riemann equation at $z_0 = x_0 + iy_0$. We can use this to show that a function is NOT differentiable at $z = z_0$.

EXAMPLE 2.17 Show that if $z \neq 0$, then $f(z) = 2xy + i(x^2 - y^2)$ is not differentiable.

Solution

Let $u = 2xy$ and $v = x^2 - y^2$. If $f(z)$ is differentiable at z, then $u_x = v_y$. But this implies that $y = 0$. Similarly, if $u_y = -v_x$ then $x = 0$. Since $z \neq 0$, we conclude that $f(z)$ is not differentiable at z.

EXAMPLE 2.18 Show that the function $f(z) = \overline{z}$ is not differentiable at any point z.

Solution

Now, $f(z) = x - iy$. Hence, $u = x$ and $v = -y$. But $u_x = 1$ and $v_y = -1$ implies that $u_x \neq v_y$ and therefore, $f(z)$ is not nowhere differentiable.

EXAMPLE 2.19 Find z for which $f'(z)$ exists when $f(z) = e^x \cos y - ie^x \sin y$.

Solution

Let $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. If $f'(z)$ exists then Cauchy-Riemann equations hold. So $u_x = v_y$ and $u_y = -v_x$ yield $\cos y = 0$ and $\sin y = 0$. But this is impossible since $\sin^2 y + \cos^2 y = 1$. Therefore, $f(z)$ is not differentiable for all $z \in \mathbf{C}$.

We return to the following statement :

If $f = u + iv$ and f is differentiable at $z = z_0$, then u and v satisfies the Cauchy-Riemann equation at $z_0 = x_0 + iy_0$.

The converse, however, is false. In other words, if u and v satisfies the Cauchy-Riemann equation at $z = z_0$, it does not imply that f is differentiable at $z = z_0$.

EXAMPLE 2.20 Consider the following example : Let

$$f(z) = f(x, y) = \begin{cases} \dfrac{xy(x+iy)}{x^2 + y^2} & z \neq 0 \\ 0 & z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations are satisfied at $z = 0$ but f is not differentiable at $z = 0$.

Solution

We first show that $f(z)$ is not differentiable at $z = 0$. Note that

$$\frac{f(z) - f(0)}{z} = \left(\frac{xy(x+iy)}{x^2 + y^2} - 0 \right) \frac{1}{x + iy} = \frac{xy}{x^2 + y^2}.$$

Letting $z = h + ih$, we find that

$$\frac{f(z) - f(0)}{z} = \frac{h^2}{2h^2} \to \frac{1}{2}$$

as $h \to 0$. Letting $z = h + i \cdot 0$, we find that

$$\frac{f(z) - f(0)}{z} \to 0$$

as $h \to 0$. Therefore

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

does not exist and f is not differentiable at $z = 0$.

We first note that

$$u = \frac{x^2 y}{x^2 + y^2} \quad \text{and} \quad v = \frac{xy^2}{x^2 + y^2}.$$

This implies that $u(h, 0) = 0$ and that

$$\frac{u(h, 0) - u(0, 0)}{h} = 0.$$

Similarly, $v(0, h) = 0$ and

$$\frac{v(0, h) - v(0, 0)}{h} = 0.$$

Therefore, $u_x(0, 0) = 0 = v_y(0, 0)$. In a similar way, $v_x(0, 0) = v_y(0, 0) = 0$. Therefore, Cauchy-Riemann equations are satisfied at $z = 0$ but f is not differentiable at $z = 0$.

THEOREM 2.3 Let $f(z) = u(x, y) + iv(x, y)$. Suppose u_x, u_y, v_x, v_y exist in a neighborhood of z. Then if u_x, u_y, v_x, v_y are continuous at z and the Cauchy-Riemann equations hold, i.e.,

$$u_x = v_y, u_y = -v_x,$$

then f is differentiable at z.

Proof

The proof of this theorem depends on mean value theorem. We recall that if F is a function of a real variable differentiable on $[a, b]$, then

$$F(b) - F(a) = F'(\xi)(b - a).$$

We next write

$$f(z) = u(x, y) + iv(x, y).$$

Then

$$f(z + \delta z) = f(x + \delta x + i(y + \delta y)) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y).$$

This implies that

$$\frac{f(z + \delta z) - f(z)}{\delta z} = \frac{u(x + \delta x, y + \delta y) - u(x, y)}{\delta x + i\delta y} + i\frac{v(x + \delta x, y + \delta y) - v(x, y)}{\delta x + i\delta y}. \tag{2.2}$$

We now apply mean value theorem to the expression on the left hand side of (2.2) which involves only $u(x, y)$ and deduce that

$$\begin{aligned}
&\frac{u(x+\delta x, y+\delta y)-u(x,y)}{\delta x+i\delta y}\\
&\qquad=\frac{u(x+\delta x, y+\delta y)-u(x, y+\delta y)+u(x, y+\delta y)-u(x,y)}{\delta x+i\delta y}\\
&\qquad=u_x(x+\theta_1, y+\delta y)\left(\frac{\delta x}{\delta x+i\delta y}\right)+u_y(x, y+\theta_2)\left(\frac{\delta y}{\delta x+i\delta y}\right),
\end{aligned}$$

where $(\theta_1, \theta_2)\to(0,0)$ as $(\delta x, \delta y)\to(0,0)$.

Similarly, applying mean value theorem to the expression on left hand side of (2.2) which involves only $v(x,y)$, we deduce that

$$\begin{aligned}
&\frac{v(x+\delta x, y+\delta y)-v(x,y)}{\delta x+i\delta y}\\
&\qquad=v_y(x, y+\theta_3)\left(\frac{\delta y}{\delta x+i\delta y}\right)+v_x(x+\theta_4, y+\delta y)\left(\frac{\delta x}{\delta x+i\delta y}\right),
\end{aligned}$$

where $(\theta_3, \theta_4)\to(0,0)$ as $(\delta x, \delta y)\to(0,0)$. Thus,

$$\begin{aligned}
\frac{f(z+\delta z)-f(z)}{\delta z}&=\frac{\delta y}{\delta x+i\delta y}\left(u_y(x, y+\theta_2)+iv_y(x, y+\theta_3)\right)\\
&\quad+\frac{\delta x}{\delta x+i\delta y}\left(u_x(x+\theta_1, y+\delta y)+iv_x(x+\theta_4, y+\delta y)\right).
\end{aligned}$$

Now, using $u_x=v_y$ and $u_y=-v_x$, we conclude that

$$\begin{aligned}
&\frac{f(z+\delta z)-f(z)}{\delta z}-(u_x+iv_x)\\
&=\frac{\delta y}{\delta x+i\delta y}\left(u_y(x, y+\theta_2)-u_y(x,y)+iv_y(x, y+\theta_3)-iv_y(x,y)\right)\\
&+\frac{\delta x}{\delta x+i\delta y}\left(u_x(x+\theta_1, y+\delta y)-u_x(x,y)+iv_x(x+\theta_4, y+\delta y)-iv_x(x,y)\right).
\end{aligned}$$

Since

$$\left|\frac{\delta x}{\delta x+i\delta y}\right|\le 1 \quad\text{and}\quad \left|\frac{\delta y}{\delta x+i\delta y}\right|\le 1,$$

we have

$$\begin{aligned}
&\left|\frac{f(z+\delta z)-f(z)}{\delta z}-(u_x+iv_x)\right|\\
&\le |(u_y(x, y+\theta_2)-u_y(x,y)|+|v_y(x, y+\theta_3)-v_y(x,y)|\\
&\qquad+|u_x(x+\theta_1, y+\delta y)-u_x(x,y)|+|v_x(x+\theta_4, y+\delta y)-v_x(x,y)|,
\end{aligned}$$

which tends to 0 as $(\delta x, \delta y)\to 0$, since $(\theta_1, \theta_2, \theta_3, \theta_4)\to(0,0,0,0)$ as $(\delta x, \delta y)\to(0,0)$ and u_x, u_y, v_x, v_y are all continuous. Hence,

$$\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}$$

exists and equals to u_x+iv_x. □

EXAMPLE 2.21 Show that $g(z) = 3x^2+2x-3y^2-1+i(6xy+2y)$ is differentiable for every $z \in \mathbf{C}$. Write $g(z)$ in terms of z.

Solution
By the above theorem, it suffices to verify that $u_x = v_y$ and $v_x = -u_y$ and that these are continuous functions. To express $g(z)$ as a function of z, set $x = (z+\overline{z})/2$ and $y = (z - \overline{z})/(2i)$. The final answer is $g(z) = 3z^2 + 2z - 1$. Note that since $g(z)$ is a polynomial, we have another proof of the fact that $g(z)$ is differentiable for every $z \in \mathbf{C}$.

2.7 Analytic functions

DEFINITION 2.5 A function f is analytic at z if f is differentiable in a neighborhood of z. A function f is analytic on a set S if f is analytic at every $z \in S$.

EXAMPLE 2.22 Polynomials are analytic on $\mathbf{C}$ since their derivatives exist at every $z \in \mathbf{C}$.

EXAMPLE 2.23 The function $f(z) = |z|^2$ is differentiable only at $z = 0$ and hence, it is not analytic at $z = 0$.

EXAMPLE 2.24 If $f = u + iv$ is analytic in a region D and u is constant, show that f is a constant.

Solution
If u is a constant, then $u_x = u_y = 0$. By Cauchy Riemann equations, $v_x = v_y = 0$. This implies that $v = s(x) = t(y)$ and hence v must also be a constant. Therefore f is a constant.

EXAMPLE 2.25 In calculus, we know that if $f'(x) = 0$ on (a, b) then $f(x)$ is a constant on (a, b). Is this true for analytic functions in $\mathbf{C}$? In other words, if $f'(z) = 0$, is $f(z)$ necessarily a constant?

Solution

The result is true. Let $f(z) = u + iv$. Suppose

$$f'(z) = u_x + iv_x = v_y - iu_y = 0.$$

This implies that $u_x = u_y = 0$ or $u = g(y) = h(x)$ for some functions g and h. Since x and y are independent variables, we find that $u = g(y) = h(x) = a$, where a is a constant in $\mathbf{R}$. By the previous example, we conclude that $f = u+iv$ is a constant.

EXAMPLE 2.26 If f is analytic in a region and if $|f|$ is constant there, show that f is a constant.

Solution

Suppose $|f|$ is a constant. If $|f| = 0$ then $u = 0$ and $v = 0$. Suppose $|f| \neq 0$. Then $|f|^2 = u^2 + v^2 = c$ where c is a constant. This implies that

$$uu_x + vv_x = 0 \tag{2.3}$$

and

$$uu_y + vv_y = 0. \tag{2.4}$$

Using the Cauchy Riemann equations $u_x = v_y$ and $u_y = -v_x$, we rewrite (2.3) and (2.3) and (2.4) as

$$uu_x - vu_y = 0 \tag{2.5}$$

and

$$uu_y + vu_x = 0. \tag{2.6}$$

Eliminating u_y using (2.5) and (2.6), we conclude that

$$u_x(u^2 + v^2) = 0.$$

Since $|f|^2$ is a non-zero constant, we find that $u_x = 0$. Solving u_y in a similar way, we conclude that $u_y = 0$. This implies that $u = s(y) = t(x)$ and so, u is a constant. By Example 2.24, we conclude that f is a constant.

2.8 The exponential function

DEFINITION 2.6 A function which is analytic on the whole of $\mathbf{C}$ is said to be entire.

EXAMPLE 2.27 Find an entire function $f(z)$ which satisfies the conditions

$$f(z+w) = f(z)f(w) \tag{2.7}$$

and

$$f(x) = e^x, \quad \text{when } x \in \mathbf{R}. \tag{2.8}$$

Solution

Let $z = x + iy$. Then by (2.7),

$$f(z) = f(x)f(iy) = e^x f(iy) = e^x(A(y) + iB(y))$$

for some functions $A(y)$ and $B(y)$.

Now, since f is analytic, Cauchy Riemann equations are satisfied and with $u = e^x A(y)$ and $v = e^x B(y)$, we have $u_x = v_y$ and $u_y = -v_x$ implies that

$$A(y) = B'(y) \quad \text{and} \quad A'(y) = -B(y)$$

respectively. This implies that

$$A''(y) + A(y) = 0.$$

Similarly

$$B''(y) + B(y) = 0.$$

Let $g(y)$ be a function satisfying

$$g''(y) + g(y) = 0.$$

It is immediate that $\sin y$ and $\cos y$ are two linearly independent solutions of the above differential equations. Since $A(y)$ and $B(y)$ are also solutions of the differential equations, we conclude that

$$A(y) = \alpha \cos y + \beta \sin y$$

and

$$B(y) = \gamma \cos y + \delta \sin y$$

for some constants α, β, γ and δ.

With the expression for $A(y)$, we deduce that

$$B(y) = -A'(y) = -\beta \cos y + \alpha \sin y.$$

Since $f(x) = e^x$, we conclude that $A(0) = \alpha = 1$. Also, $B(0) = 0 = -\beta$. Therefore,

$$A(y) = \cos y \quad \text{and} \quad B(y) = \sin y.$$

Hence,

$$f(z) = e^x (\cos y + i \sin y).$$

The function f extends e^x and is entire. We write $f(z)$ as e^z.

Note that if we compare $e^z = e^{x+iy}$ with $e^x(\cos y + i\sin y)$, then we see that

$$e^{iy} = \cos y + i \sin y.$$

Therefore,

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \quad \text{and} \quad \cos y = \frac{e^{iy} + e^{-iy}}{2}.$$

As in the case of e^x, we may define sine and cosine with variables $z \in \mathbf{C}$ as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Remark 2.4 We have assumed that $A(y)$ and $B(y)$ are linear combinations of $\sin y$ and $\cos y$ and that the expressions for $A(y)$ and $B(y)$ are unique. For more information on the uniqueness of solutions of differential equations with initial conditions, see Chapter 3 (pp. 62-80) of E.L.Ince's "Ordinary differential equations".

EXAMPLE 2.28 Show that

$$\frac{de^z}{dz} = e^z, \frac{d\sin z}{dz} = \cos z \quad \text{and} \quad \sin(z+w) = \sin z \cos w + \sin w \cos z.$$

2.9 Harmonic functions

Let $f(z) = u + iv$ be an analytic function on a region R. We will show later that $f'(z)$ is also an analytic function on the region R. Suppose for the moment, we assume that

If f is analytic at z, then $f'(z)$ is analytic at z.

We observe that

$$f'(z) = u_x + iv_x = v_y - iu_y = \Phi + i\Psi. \tag{2.9}$$

Let $\Phi = u_x$ and $\Psi = -u_y$. Since f' is analytic $\Phi_x = \Psi_y$. Hence,

$$u_{xx} = -u_{yy}$$

and u is harmonic. Similarly, using $\Phi = v_y$ and $\Psi = v_x$ and the fact that $\Phi_y = -\Psi_x$, we deduce that v is harmonic.

DEFINITION 2.7 A real-valued function $h(x, y)$ is said to be harmonic in a region R if all its second order partial derivatives are continuous in R and if, at each point of R,

$$h_{xx}(x, y) + h_{yy}(x, y) = 0.$$

We sometimes use $\nabla^2 h$ or Δh to denote $h_{xx} + h_{yy}$.

We have seen in the beginning of this section that if f is analytic at z then u and v are harmonic functions.

DEFINITION 2.8 Let u be harmonic. If v is a harmonic function satisfying the Cauchy-Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x,$$

then v is called the conjugate harmonic function of u.

EXAMPLE 2.29 Verify that $u = xy - x + y$ is a harmonic function and find its harmonic conjugate.

Solution

The verification that u is a harmonic function is straightforward since $u_{xx} = u_{yy} = 0$. Now, $u_x = y - 1$ and $u_y = x + 1$. Using Cauchy-Riemann equations, we find that $v_y = y - 1$ or $v = y^2/2 - y + s(x)$. Also, $-v_x = x + 1$ implies that $v = -x^2/2 - x + t(y)$. Now, we must have

$$-\frac{x^2}{2} - x - s(x) = \frac{y^2}{2} - y - t(y) = C$$

where C is a constant. Hence,

$$t(y) = \frac{y^2}{2} - y - C$$

and

$$v = -\frac{x^2}{2} - x + \frac{y^2}{2} - y - C.$$

EXAMPLE 2.30 Show that if $f = u + iv$ is analytic, then $u + v$ is harmonic.

Solution
First solution :

$$(u+v)_{xx} + (u+v)_{yy} = u_{xx} + v_{xx} + u_{yy} + v_{yy} = 0.$$

Second Solution:
Note that $f = u + iv$ and $if = v - iu$ are both analytic. Therefore $f + if = (u+v) + i(v-u)$ is analytic and hence $u + v$ is harmonic.

2.10 Appendix

We motivate the general definition of continuous functions by proving the following :

THEOREM 2.4 Let f be a surjective function from $\mathbf{C}$ to $\mathbf{C}$. The following are equivalent :

(a) f is continuous
(b) If V is an open set in $\mathbf{C}$, then $f^{-1}(V)$ is an open set in $\mathbf{C}$, where

$$f^{-1}(A) = \{z \in \mathbf{C} \,|\, f(z) \in A\}.$$

Proof
We first show (a) implies (b). We translate continuity of f in terms of open balls. Observe that f is continuous at z_0 if and only if for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$f(B(z_0;\delta_\epsilon)) \subset B(f(z_0);\epsilon).$$

Now, let V be open in $\mathbf{C}$. We must show that $f^{-1}(V)$ is open in $\mathbf{C}$. Let $z_0 \in f^{-1}(V)$. Then $f(z_0) \in V$. Since V is open, there exists $\epsilon > 0$ such that

$$B(f(z_0);\epsilon) \subset V.$$

For this $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$f(B(z_0;\delta_\epsilon)) \subset B(f(z_0);\epsilon) \subset V,$$

by continuity of f. This means that

$$B(z_0;\delta_\epsilon) \subset f^{-1}(B(f(z_0);\epsilon)) \subset f^{-1}(V).$$

Hence $f^{-1}(V)$ is open.
To show that (b) implies (a). Let f be a function satisfying (b). Let $\epsilon > 0$

and since $B(f(z_0);\epsilon)$ is open, $f^{-1}(B(f(z_0);\epsilon))$ is open in the domain of f. Since $z_0 \in f^{-1}(B(f(z_0);\epsilon))$ and the set is open, there exists $\delta_\epsilon > 0$ such that

$$B(z_0;\delta_\epsilon) \subset f^{-1}(B(f(z_0);\epsilon)),$$

or

$$f(B(z_0;\delta_\epsilon)) \subset B(f(z_0);\epsilon),$$

and so f is continuous. □

The usual definition for continuous function from X to Y where X and Y are topological spaces is the following:

DEFINITION 2.9 A function f from topological spaces X to Y is continuous if for every open set V in Y, the set $f^{-1}(V)$ is open in X.

If X and Y are subsets of $\mathbf{C}$, we can form topological spaces from X and Y by declaring that the open sets in X and Y are sets of the form $O \cap X$ and $O \cap Y$ respectively, with O open in $\mathbf{C}$. We sometimes call such open sets relatively open sets. I prefer to call sets $O \cap X$ (or $O \cap Y$), O open in $\mathbf{C}$, as sets that are "open in X" (or "open in Y").

We now show the following:

THEOREM 2.5 If f is a continuous function from X to Y (in the usual sense), then for every V open in Y, $f^{-1}(V)$ is open in X.

Proof
The function f is continuous at u means that for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$f(B(u;\delta_\epsilon)) \subset B(f(u);\epsilon).$$

Suppose V is open in Y. Then $V = O \cap Y$ where O is an open set of $\mathbf{C}$. Now

$$f^{-1}(V) = f^{-1}(O) \cap X.$$

Let $u \in f^{-1}(V)$. Then $f(u) \in V$ and hence, $f(u) \in O$. This implies that there exists $\epsilon > 0$ such that

$$B(f(u);\epsilon) \cap Y \subset O \cap Y$$

since O is open. Since f is continuous at u, there exists $\delta(u) > 0$ such that

$$f(B(u;\delta(u)) \cap X) \subset B(f(u);\epsilon) \cap Y.$$

Hence

$$B(u;\delta) \cap X \subset f^{-1}(B(f(u);\epsilon)) \cap X \subset f^{-1}(V).$$

Therefore,

$$f^{-1}(V) = \bigcup_{u \in f^{-1}(V)} (B(u;\delta(u)) \cap X) = \left(\bigcup_{u \in f^{-1}(V)} B(u;\delta) \right) \cap X$$

and this implies that $f^{-1}(V)$ is open. $\square$

3 Line Integrals

In real analysis, we encounter integrals defined by

$$\int_a^b f(x)dx$$

for some interval (a, b). In complex analysis, we have analogue of integrals as well. We will begin with the simplest type of complex integrals, the line integrals.

3.1 Properties of Line integrals

DEFINITION 3.1 Let $f(t) = u(t) + iv(t)$ be any continuous complex valued function of the real variable t, $a \leq t \leq b$. Then

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

Note that in general, our $f(z)$ depends on two variables x, y since $z = x + iy$. Here, we assume that z is of the form $x(t) + iy(t)$ and consequently, $f(z)$ is a function of t.

DEFINITION 3.2 If $x(t)$, $y(t)$ are continuous on $[a, b]$ and their derivatives $x'(t)$, $y'(t)$ are continuous on intervals $[a, x_1], [x_1, x_2], \cdots, [x_{n-1}, b]$, where

$$[a, b] = \bigcup_{i=0}^{n-1} [x_i, x_{i+1}], x_0 = a, x_n = b,$$

then we say that the curve

$$z(t) = x(t) + iy(t)$$

is piecewise differentiable and we set

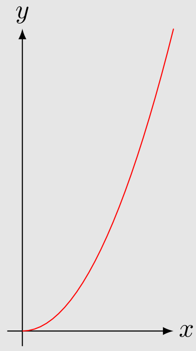
$$\dot{z}(t) = x'(t) + iy'(t).$$

DEFINITION 3.3 The curve is said to be smooth if $\dot{z}(t) \neq 0$ except at a finite number of points.

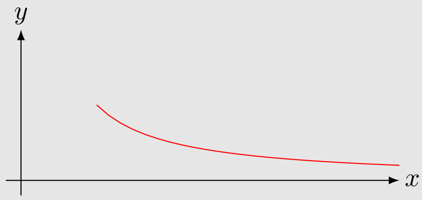
We will assume smoothness throughout our course unless otherwise stated.

EXAMPLE 3.1 Examples of smooth curves:

(i) $z(t) = t + it^2, 0 \leq t \leq 2.$



(ii) $z(t) = t + \dfrac{i}{t}, 1 \leq t \leq 5.$



DEFINITION 3.4 Let C be a smooth curve given by $z(t), a \leq t \leq b$, and suppose f is continuous at all points $z(t)$. Then the integral of f along C is defined by

$$\int_C f(z)dz := \int_a^b f(z(t))\dot{z}(t)dt.$$

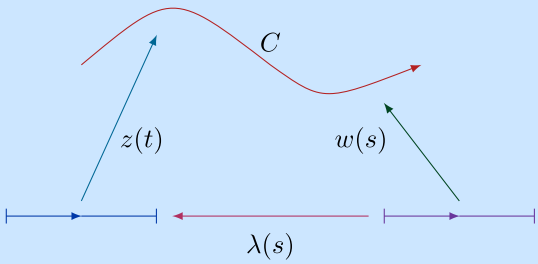
DEFINITION 3.5 The two curves

$$C_1 : z(t), a \leq t \leq b,$$

$$C_2 : w(s), c \leq s \leq d,$$

are smoothly equivalent if there exist a one to one function $\lambda : [c, d] \to [a, b]$ which has continuous first derivative such that $\lambda(c) = a$, $\lambda(d) = b$ and

$$w(s) = z(\lambda(s)).$$



EXAMPLE 3.2 The functions

$$z(t) = t + it^2, 0 \leq t \leq 3$$

and

$$w(s) = (s-1) + i(s-1)^2, 1 \leq s \leq 4$$

give the same curve. Here, $\lambda(s) = s - 1$.

THEOREM 3.1 If C_1 and C_2 are smoothly equivalent then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Proof
Suppose C_1 is given by $z(t)$, $t \in [a, b]$ and C_2 is given by $w(s)$, $s \in [c, d]$, with $\lambda(s)$ being a one to one function with continuous first derivative from $[c, d]$ to $[a, b]$ and

$$w(s) = z(\lambda(s)).$$

Now,

$$\int_{C_2} f(w)dw = \int_c^d f(w(s))\dot{w}(s)ds$$

$$= \int_c^d f(z(\lambda(s)))\dot{z}(\lambda(s))\lambda'(s)ds$$
$$= \int_a^b f(z(\mu))\dot{z}(\mu)d\mu, \quad \text{with } \mu = \lambda(s)$$
$$= \int_{C_1} f(z)dz.$$

Hence the result. □

Remark 3.1 The second equality above follows by splitting the second integral into real and complex part, apply integral by substitution, and collect the resulting integrals. We write the proof as above to show that the result follows from the substitution $\mu = \lambda(s)$.

DEFINITION 3.6 Suppose C is given by $z(t), a \leq t \leq b$. Then $-C$ is defined by $z(b+a-s), a \leq s \leq b$.

THEOREM 3.2 Let C be a smooth curve and f be a continuous complex-valued function. Then

$$\int_{-C} f(z)dz = -\int_C f(z)dz.$$

Proof
Let $w(s) = z(b+a-s)$. Then

$$\int_{-C} f(w)\, dw = \int_a^b f(w(s))\dot{w}(s)ds = \int_a^b f(z(b+a-s))\dot{z}(b+a-s)(b+a-s)'\, ds$$
$$= \int_b^a f(z(t))\dot{z}(t)(-1)(-1)dt,$$

where we let $t = b+a-s$ and observe that $ds = -dt$. Therefore,

$$\int_{-C} f(w)\, dw = -\int_a^b f(z(t))\dot{z}(t)dt = -\int_C f(z)dz.$$

□

THEOREM 3.3 Let C be a smooth curve. Let f and g be continuous functions on C, and let α be any complex number. Then

(a) $\int_C f(z) + g(z)dz = \int_C f(z)dz + \int_C g(z)dz$,

(b) $\int_C \alpha f(z)dz = \alpha \int_C f(z)dz$.

EXAMPLE 3.3 Set $C : z(t) = R\cos t + iR\sin t, 0 \le t \le 2\pi, R \neq 0$. Show that

$$\int_C \frac{1}{z}\, dz = 2\pi i.$$

Solution

The curve C is parametrized by $z(t) = R\cos t + iR\sin t$. This implies that $\dot{z}(t) = -R\sin t + R\cos t$. Hence

$$\begin{aligned}\int_C \frac{1}{z} dz &= \int_0^{2\pi} \frac{1}{R(\cos t + i\sin t)} (R(-\sin t + i\cos t))\, dt \\ &= \int_0^{2\pi} \frac{(-\sin t + i\cos t)(\cos t - i\sin t)}{\sin^2 t + \cos^2 t}\, dt \\ &= \int_0^{2\pi} i\, dt = 2\pi i.\end{aligned}$$

Note that the above integral is independent of the radius R of the contour C.

EXAMPLE 3.4 Suppose $f(z) = x^2 + iy^2$, and Let $C : z(t) = t + it^2, 0 \le t \le 1$. Compute

$$\int_C x^2 + iy^2\, dz.$$

Solution

The curve C is parametrized by $C : z(t) = t + it^2, 0 \le t \le 1$. Then $\dot{z}(t) = 1 + 2it$.

$$\int_C f(z) dz = \int_0^1 (t^2 + it^4)(1 + 2it) dt = \frac{7i}{10}.$$

The line integral of $f(z)$ over C has properties similar to Riemann integrals. We state two of them here and leave the proofs as exercises.

EXAMPLE 3.5 Evaluate

$$\int_i^{i/2} e^{\pi z} dz$$

where this integral denotes the path (or contour) integral over the straight line from the point i to $i/2$.

Solution
The integral is evaluated as follow:

$$\begin{aligned}
\int_i^{i/2} e^{\pi z} dz &= \int_0^1 e^{\pi((1-t)i+ti/2)} \left(-i + i/2\right) dt \\
&= -\frac{i}{2}\int_0^1 e^{\pi i(1-t/2)} dt \\
&= \frac{i}{2}\int_0^1 e^{-\pi i t/2} dt \\
&= \frac{i}{2(-i\pi/2)}\left(e^{-i\pi/2} - 1\right) \\
&= \frac{1}{\pi}\left(1+i\right).
\end{aligned}$$

EXAMPLE 3.6 Show that if $n \neq -1$ and C is a circle with center 0 and radius R traversed in the anti-clockwise direction, then

$$\int_C z^n dz = 0.$$

Solution
Let C be parametrized by $z(t) = Re^{it}, 0 \le t \le 2\pi$. Then

$$\begin{aligned}
\int_C z^n dz &= \int_0^{2\pi} R^n e^{int} i R e^{it} dt \\
&= iR^{n+1}\int_0^{2\pi} e^{i(n+1)t} dt \\
&= iR^{n+1}\left(\int_0^{2\pi} \cos((n+1)t) dt + i\int_0^{2\pi} \sin((n+1)t) dt\right) = 0.
\end{aligned}$$

3.2 A generalization of the Fundamental Theorem of Calculus

We begin with two examples.

EXAMPLE 3.7 Show that

$$\int_C z dz = 0$$

by taking C to be

(a) the upper semicircle (of radius 1, center 0) from -1 to 1 in the clockwise direction.
(b) the lower semicircle (of radius 1, center 0) from -1 to 1 in the anti-clockwise direction.

Solution

For (a), we note that

$$\int_C z dz = \int_\pi^0 e^{it} i e^{it} dt = 0.$$

For (b), we find that

$$\int_C z dz = \int_{-\pi}^0 e^{it} i e^{it} dt = 0.$$

This example shows that we get the same answer if we change the path joining i and $i/2$. In fact, this integral depends only on the end points.

EXAMPLE 3.8 Show that

$$\int_C \frac{1}{z} dz = 0$$

by taking C to be

(a) the upper semicircle (of radius 1, center 0) from -1 to 1 in the clockwise direction.
(b) the lower semicircle (of radius 1, center 0) from -1 to 1 in the anti-clockwise direction.

Solution

For (a), we note that

$$\int_C \frac{1}{z} dz = \int_\pi^0 e^{-it} i e^{it} dt = i(0 - \pi) = -i\pi.$$

For (b), we find that

$$\int_C \frac{1}{z} dz = \int_{-\pi}^0 e^{-it} i e^{it} dt = i(0 - (-\pi)) = i\pi.$$

This example shows that the integral is dependent on the path joining i and $i/2$.

In the first example, the integral is the same for (a) and (b) while the second example shows that there are integral that depends on paths. In the following theorem, we show the existence of integrals that only depend on the points of a path and not on the paths joining them.

DEFINITION 3.7 A primitive for f on a region D is a function F that is analytic on D and such that $F'(z) = f(z)$ for all $z \in D$.

THEOREM 3.4 If a continuous function f has a primitive F in a region D and C is a smooth curve that begins at α and ends at β, then

$$\int_C f(z)dz = F(\beta) - F(\alpha).$$

We emphasize that the above theorem implies that if $f(z) = F'(z)$ in a region D where $F(z)$ is analytic in D and if C is any smooth curve from α to β contained in D, then the integral $\int_C f(z)dz$ depends only on end points α and β.

Proof
The proof depends on a complex analogue of the chain rule for differentiation. We first assume that C is smooth and parametrized by $z(t)$, $a \leq t \leq b$, with $z(a) = \alpha$ and $z(b) = \beta$. We observe that there exists a $\delta > 0$ such that $z(t+h) - z(t)$ is non-zero whenever $|h| < \delta$. This is because $\dot{z}(t) \neq 0$ implies that $|\dot{z}(t)| > 0$. Let $\epsilon = |\dot{z}(t)|/2$. Then there exists a $\delta > 0$ such that

$$\left| \frac{z(t+h) - z(t)}{h} - \dot{z}(t) \right| < |\dot{z}(t)|/2$$

whenever $|h| \leq \delta$. Using the triangle inequality, we deduce that

$$|\dot{z}(t)| - |(z(t+h) - z(h))/h| < |\dot{z}(t)|/2,$$

or

$$|z(t+h) - z(t)| > h|\dot{z}(t)|/2 > 0.$$

Let $\gamma(t) = F(z(t))$. Then since $z(t+h) - z(t)$ is non-zero for $|h| < \delta$, we find that

$$\begin{aligned} \dot{\gamma}(t) &= \lim_{h \to 0} \frac{F(z(t+h)) - F(z(t))}{h} \\ &= \lim_{h \to 0} \frac{F(z(t+h)) - F(z(t))}{z(t+h) - z(t)} \cdot \frac{z(t+h) - z(t)}{h}. \end{aligned}$$

Thus,

$$\dot{\gamma}(t) = f(z(t))\dot{z}(t).$$

Now, observe that

$$\begin{aligned} \int_C f(z)\, dz &= \int_a^b f(z(t))\dot{z}(t)\, dt = \int_a^b \dot{\gamma}(t)\, dt \\ &= \gamma(b) - \gamma(a) = F(z(b)) - F(z(a)) = F(\beta) - F(\alpha) \end{aligned}$$

and the proof of the theorem is complete.

□

Remark 3.2 If C is only piecewise smooth from α to β, it is a union of finitely many piecewise smooth curves. We may apply the above result to each of these smooth segment and the integral over piecewise smooth curve is still dependent only on α and β.

EXAMPLE 3.9 By finding a primitive for $f(z)$, evaluate

$$\int_{[i,i/2]} e^{\pi z} dz$$

where $[i, i/2]$ denote the straight path from i to $i/2$.

Solution

It is known that if $F(z) = \dfrac{e^{\pi z}}{\pi}$, then (this is obtained by the integrating $e^{\pi x}$ when x is real)

$$F'(z) = e^{\pi z}.$$

Hence,

$$\int_{[i,i/2]} e^{\pi z} dz = \frac{1}{\pi}\left(e^{\pi i/2} - e^{\pi i}\right).$$

Remark 3.3 The integral $\int_{C(0;R)} z^n dz$ is always 0 when $n \neq -1$ because

$$\frac{d(z^{n+1}/(n+1))}{dz} = z^n$$

and $z^{n+1}/(n+1)$ is analytic on $C(0;R)$.

3.3 The ML-formula

LEMMA 3.5 Suppose $G(t)$ is a continuous complex valued function of t. Then

$$\left|\int_a^b G(t)dt\right| \leq \int_a^b |G(t)|\, dt.$$

Proof
Let
$$\int_a^b G(t)dt = Re^{i\theta}, R \geq 0,$$
since the integral is a complex number. Therefore,
$$\int_a^b e^{-i\theta} G(t)dt = R.$$
Suppose
$$e^{-i\theta} G(t) = A(t) + iB(t),$$
where $A(t)$ and $B(t)$ are real valued functions of t. Then
$$R = \int_a^b A(t)dt + i \int_a^b B(t)dt.$$
But since R is real,
$$\int_a^b B(t)dt = 0$$
and
$$R = \int_a^b A(t)dt = \int_a^b \text{Re}\left(e^{-i\theta} G(t)\right) dt.$$
But $|\text{Re}\,(z)| < |z|$ and so,
$$R \leq \int_a^b \left|e^{-i\theta} G(t)\right| dt = \int_a^b |G(t)|\, dt$$
since $|e^{i\theta}| = 1$. Therefore,
$$\left| \int_a^b G(t)dt \right| = R|e^{i\theta}| = R \leq \int_a^b |G(t)|dt,$$
or
$$\left| \int_a^b G(t)dt \right| \leq \int_a^b |G(t)|dt.$$
□

We now establish the ML-formula.

THEOREM 3.6 Let C be a smooth curve of length L and f be continuous on C and $|f| \leq M$ on C. Then
$$\left| \int_C f(z)dz \right| \leq ML.$$

Proof
Let C be represented by $z(t) = x(t) + iy(t)$. Then by Lemma 3.5,

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))\dot{z}(t)dt\right| \leq \int_a^b |f(z(t))| \cdot |\dot{z}(t)|\, dt.$$

Now,

$$\begin{aligned}\int_a^b |f(z(t))| \cdot |\dot{z}(t)|\, dt &\leq \int_a^b \max_{z\in C}(|f(z(t))|)|\dot{z}(t)|dt \\ &\leq M \int_a^b |\dot{z}(t)|dt.\end{aligned}$$

Next recall that if a curve $z(t) = x(t) + iy(t)$ then the length L of the curve is given by [1]

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_a^b |\dot{z}(t)|dt.$$

Hence,

$$\left|\int_C f(z)dz\right| \leq ML.$$

□

EXAMPLE 3.10 Let C be the contour $z = 3e^{i\theta}, 0 \leq \theta \leq \pi$. Show that

$$\left|\int_C \frac{z}{z^2+1}dz\right| \leq \frac{9\pi}{8}.$$

Solution
From Theorem 3.6, we know that

$$\left|\int_C \frac{z}{z^2+1}dz\right| \leq ML,$$

where M is the bound of $|f|$ on C and L is the length of C. We now compute M. On C, $z = 3e^{i\theta}$ and so, $|z| = 3$ and $|z^2 + 1| \geq 9 - 1 = 8$. This implies that

$$\frac{1}{|z^2+1|} \leq \frac{1}{8}.$$

Hence,

$$\left|\frac{z}{z^2+1}\right| \leq \frac{3}{8},$$

on C. Therefore $M \leq \frac{3}{8}$. Next, the length L of the arc C is clearly 3π since this

[1] Stewart's Calculus, Theorem 9.1.2.

is the perimeter of the semi-circle with radius 3. Therefore, by the ML-formula, we deduce that

$$\left| \int_C \frac{z}{z^2+1} dz \right| \le \frac{9\pi}{8}.$$

4 The Cauchy-Goursat Theorem

4.1 The Cauchy-Goursat Theorem

DEFINITION 4.1 A closed curve C is a curve where the initial point and terminal point meets. A simple closed curve C is a closed curve which has no other meeting points.

DEFINITION 4.2 A set S is star-shaped if it has a point s, called the star center, so that for each $z \in S$, the segment $[s, z]$ lies in S. Here $[s, z]$ denote the line joining s and z.

Remark 4.1 Suppose $P(x, y)$ and $Q(x, y)$ together with their partial derivatives are continuous in the region bounded by a simple closed curve. Then according to Green's Theorem in advanced calculus

$$\int_C Pdx + Qdy = \iint_R (Q_x - P_y)dxdy.$$

The first integral is over the contour and the second is over the region bounded by the contour.

Now consider

$$f(z) = u(x, y) + iv(x, y),$$

is analytic in R bounded by C. Then

$$\begin{aligned}\int_C f(z)\,dz &= \int_C (u + iv)(dx + idy)\\ &= \int_C udx - vdy + i(\int_C vdx + udy)\\ &= \iint_R (-v)_x - u_y dxdy + i(\iint_R u_x - v_y dxdy) = 0,\end{aligned}$$

since Cauchy-Riemann equations are satisfied, namely, $u_x = v_y$ and $u_y = -v_x$. Therefore,

$$\int_C f(z)dz = 0.$$

In the above discussion, we have assumed that f' is continuous. Our aim will be to prove a result similar to the above without assuming the continuity of f'. We will now remove this condition and prove the following Theorem due to Goursat:

THEOREM 4.1 Suppose f is analytic on a star-shaped region S and C is a simple closed curve in S traversed in the counterclockwise direction. Then

$$\int_C f(z)dz = 0.$$

Before we proceed to the proof, we look at some examples.

EXAMPLE 4.1 Let $f(z) = ze^{-z}$. Now ze^{-z} is analytic in $|z| \leq 1$. Therefore

$$\int_C f(z)dz = 0.$$

DEFINITION 4.3 Let $d(x, y) = |x - y|$ and if S is a set in $\mathbf{C}$, we let the diameter of S, denoted by diam S, to be

$$\operatorname{diam} S = \sup\{d(x, y)|x, y \in S\}.$$

LEMMA 4.2 Suppose $\{F_k\}$ is a collection of non-empty closed sets with

$$F_1 \supset F_2 \supset F_3 \supset \cdots$$

and diam $F_n \to 0$ as $n \to \infty$, then $\displaystyle\bigcap_{n=1}^{\infty} F_n$ consists of a single point.

A sequence $\{z_k\} \subset S$ is Cauchy if for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbf{Z}^+$ such that $|z_n - z_m| < \epsilon$ whenever $n > m > N_\epsilon$. If S is a set for which every Cauchy sequences in S is convergent, then we say that S is complete. It is known that $\mathbf{R}$ is complete. Note that this is a consequence of the axiom that every sequence that is bounded above has a least upper bound. The fact that **R** is complete implies that **C** is complete. In other words, every Cauchy sequence in **C** is convergent.

Proof

Let $z_k \in F_k$. Since diam $F_n \to 0$ as $n \to \infty$, we find that for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbf{Z}^+$ such that

$$\operatorname{diam} F_n < \epsilon$$

whenever $n \geq N_\epsilon$. Let $n > m > N_\epsilon$. Then $z_n, z_m \in F_{N_\epsilon}$ and

$$|z_n - z_m| \leq \operatorname{diam} F_{N_\epsilon} < \epsilon.$$

This implies that $\{z_k\}$ is a Cauchy sequence and therefore it converges to a limit z_0. If

$$z_0 \in \bigcap_{n=1}^{\infty} F_n,$$

then we have proved that $\bigcap_{n=1}^{\infty} F_n$ is non-empty.

If

$$z_0 \notin \bigcap_{n=1}^{\infty} F_n,$$

then

$$z_0 \in \bigcup_{n=1}^{\infty} F_n^c.$$

This implies that $z_0 \in F_s^c$ for some integer $s \in \mathbf{Z}^+$. Since F_s is closed, F_s^c is open and $B(z_0; \delta) \subset F_s^c$ for some $\delta > 0$. This implies that

$$B(z_0; \delta) \cap F_s = \phi,$$

or

$$|w - z_0| \geq \delta$$

for all $w \in F_s$. If $n > s$, then F_s contains F_n and hence $|z_n - z_0| \geq \delta$ since $z_n \in F_n \subset F_s$. This implies that z_n does not converge to z_0.

Now, suppose $z_0, y \in \bigcap_{n=1}^{\infty} F_n$. Then

$$d(z_0, y) \leq \operatorname{diam}(F_n),$$

for $n \geq 1$. But $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$ implies that $d(z_0, y) = 0$. Hence $z_0 = y$. This implies that

$$\bigcap_{n=1}^{\infty} F_n = \{z_0\}.$$

□

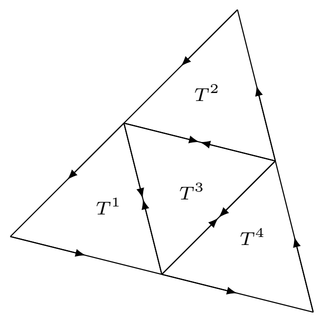
LEMMA 4.3 If f is analytic on an open set D and T is a closed triangular region with boundary ∂T that lies in D, then

$$\int_{\partial T} f(z)dz = 0.$$

Proof
We will prove the result by contradiction. Suppose

$$I := \int_{\partial T} f(z)dz \neq 0.$$

Take the mid-point of each side and subdivide T into four triangles T^1, T^2, T^3, and T^4 (See the following diagram).



Note that

$$\int_{\partial T} f(z)dz = \sum_{k=1}^{4} \int_{\partial T^k} f(z)dz.$$

Hence, by triangle inequality,

$$0 \neq \left| \int_{\partial T} f(z)dz \right| \leq \sum_{k=1}^{4} \left| \int_{\partial T^k} f(z)dz \right|. \tag{4.1}$$

If

$$\left| \int_{\partial T^k} f(z)dz \right| < \frac{1}{4} \left| \int_{\partial T} f(z)dz \right| \tag{4.2}$$

for $k = 1, 2, 3, 4$, then by (4.1), we conclude that

$$|I| < \frac{1}{4} \cdot 4 \cdot |I| = |I|,$$

which is a contradiction if $|I| > 0$. This implies that (4.2) cannot be true and there exists i_0 with $1 \leq i_0 \leq 4$ such that

$$\left| \int_{\partial T^{i_0}} f(z)dz \right| \geq \frac{1}{4} \left| \int_{\partial T} f(z)dz \right|. \tag{4.3}$$

We now rename $T_0 = T$ and $T_1 = T^{i_0}$.

Note that the diameter of T^{i_0} is half the diameter of T_0, i.e.,

$$\mathrm{diam}(T^{i_0}) = \frac{\mathrm{diam}(T_0)}{2},$$

and the length of ∂T^{i_0} is also half of the length of ∂T_0, or,

$$L(\partial T^{i_0}) = \frac{L(\partial T_0)}{2}. \tag{4.4}$$

Next, we repeat the above process with T_0 replaced by T_1 and so on. We then obtain a sequence of triangles $T_1, T_2 = T_1^{i_1}, \cdots, T_n = T_{n-1}^{i_{n-1}}$ such that the diameter of T_n is $1/2$ the diameter of T_{n-1}, which by induction, is $1/2^n$ times the diameter of T_0, namely,

$$\mathrm{diam}(T_n) = \frac{\mathrm{diam}(T_0)}{2^n}, \tag{4.5}$$

the length of ∂T_n is $1/2^n$ times the length of T, or

$$L(\partial T_n) = \frac{L(\partial T)}{2^n}.$$

and that

$$\left|\int_{\partial T_n} f(z)dz\right| \geq \frac{1}{4^n}\left|\int_{\partial T} f(z)dz\right|.$$

Note that $\{T_n\}$ is a sequence of closed sets satisfying $T_0 \supset T_1 \supset T_2 \supset \cdots$ with $\mathrm{diam}(T_n) \to 0$ as $n \to \infty$ by (4.5). By Lemma 4.2, there must be a point

$$z_0 \in \bigcap_{n=0}^{\infty} T_n.$$

Let

$$\epsilon = \frac{|I|}{2\mathrm{diam}(T_0)L(\partial T_0)} > 0.$$

Since f is differentiable at z_0, there is a $\delta_\epsilon > 0$ so that

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon,$$

whenever $0 < |z - z_0| < \delta_\epsilon$. In other words,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon|z - z_0|. \tag{4.6}$$

We next claim that there is a positive integer N such that $T_N \subset B(z_0; \delta_\epsilon)$. To prove this claim, we observe that since diam $T_n \to 0$ as $n \to \infty$, there exists $N \in Z^+$ such that

$$\mathrm{diam}\, T_n < \frac{\delta_\epsilon}{2}$$

whenever $n \geq N$. Since $z_0 \in T_N$, this implies that for all $u \in T_N$,

$$|u - z_0| < \operatorname{diam} T_N < \frac{\delta_\epsilon}{2} < \delta_\epsilon.$$

In other words,

$$T_N \subset B(z_0; \delta_\epsilon).$$

Combining this claim with (4.6), we deduce that for all $z \in T_N$,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon |z - z_0|. \tag{4.7}$$

Now, both $f'(z_0)(z-z_0)$ and $f(z_0)$ have primitives which are $f'(z_0)(z^2/2 - z_0 z)$ and $f(z_0)z$, respectively. Hence,

$$\int_{\partial T_N} f'(z_0)(z - z_0)dz = 0 \quad \text{and} \quad \int_{\partial T_N} f(z_0)dz = 0$$

and we deduce that

$$\int_{\partial T_N} (f(z) - f(z_0) - f'(z_0)(z - z_0))dz = \int_{\partial T_N} f(z)dz.$$

Hence, we find that

$$\begin{aligned} 0 < \frac{1}{4^N}|I| = \frac{1}{4^N}\left| \left|\int_{\partial T_0} f(z)dz\right| \right. &\leq \left|\int_{\partial T_N} f(z)dz\right| \\ = \left|\int_{\partial T_N} f(z) - f(z_0) - f'(z_0)(z - z_0)dz\right|&. \end{aligned} \tag{4.8}$$

Since we are integrating over ∂T_N, we deduce from (4.7) that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon |z - z_0| < \epsilon \cdot \operatorname{diam}(T_N). \tag{4.9}$$

Using (4.8),(4.9), (4.5), (4.4) and the ML-formula, we deduce that

$$\begin{aligned} 0 < |I| = \left|\int_{\partial T_0} f(z)dz\right| &< 4^N \epsilon (\operatorname{diam} T_N) L(\partial T_N) \\ &\leq 4^N \epsilon \frac{1}{2^N} \operatorname{diam}(T_0) \frac{1}{2^N} L(\partial T_0) = \frac{|I|}{2}. \end{aligned}$$

This is clearly a contradiction and we conclude that $I = 0$. □

4.2 Existence of primitive

Recall in calculus that if f is continuous on (a, b), we can create a primitive by letting

$$F(x) = \int_{x_0}^{x} f(t)\, dt.$$

We then show that $F'(x) = f(x)$. As an application, we define

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

and note that

$$\frac{d \ln x}{dx} = \frac{1}{x}.$$

In order to create primitive for continuous function on star-shaped region, we will follow the above idea and set

$$F(z) = \int_{[s,z]} f(\zeta)\, d\zeta$$

where s is the star center. This is the starting point of the proof of the following theorem.

THEOREM 4.4 Let S be an open star-shaped region and f be continuous on S. Let T be a closed triangular region and ∂T be the boundary of the triangle traversed in the anticlockwise direction. Suppose

$$\int_{\partial T} f(z) dz = 0$$

for every T in S, then f has a primitive F on S.

Proof
Let s be a star center for S. For each $z \in S$, define

$$F(z) = \int_{[s,z]} f(\zeta) d\zeta.$$

We will show that

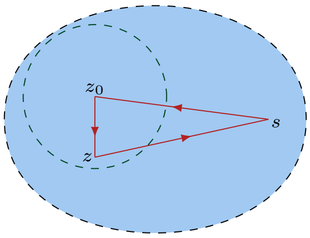
$$F'(z_0) = f(z_0)$$

for all $z_0 \in S$.

Let $z_0 \in S$. Since S is open, there exist a $\delta_1 > 0$ such that $B(z_0; \delta_1) \subset S$ (S is open).

Next, since f is continuous at z_0, we find that for $\epsilon > 0$, there exists $\delta_2 > 0$ such that

$$|f(\zeta) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta_2 \tag{4.10}$$



Let $\delta = \min(\delta_1, \delta_2)$. For $z \in B(z_0; \delta)$,

$$\int_{[z_0,z]} + \int_{[s,z_0]} + \int_{[z,s]} f(\zeta)d\zeta = 0,$$

since by hypothesis, the integral over any boundary of a triangle in S is 0. This implies that

$$\begin{aligned}\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) &= \frac{1}{z-z_0}\left\{\int_{[s,z]} f(\zeta)d\zeta - \int_{[s,z_0]} f(\zeta)d\zeta - (z-z_0)f(z_0)\right\}\\ &= \frac{1}{z-z_0}\left\{\int_{[z_0,z]} f(\zeta)d\zeta\right\} - \frac{f(z_0)}{z-z_0}\int_{[z_0,z]} 1d\zeta\\ &= \frac{1}{z-z_0}\int_{[z,z_0]} \left(f(z_0)-f(\zeta)\right)d\zeta.\end{aligned}$$

Therefore, for $0 < |z-z_0| < \delta$,

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| < \frac{1}{|z-z_0|}\epsilon|z-z_0| = \epsilon,$$

where we have used (4.10) and the ML-formula. This implies that $F'(z_0) = f(z_0)$. □

We now state the Cauchy-Goursat Theorem for star shaped region.

THEOREM 4.5 Suppose f is analytic on a star-shaped region S. Then for every simple closed path C in S traversed in the counterclockwise direction,

$$\int_C f(z)dz = 0.$$

Proof
By Lemma 4.3,

$$\int_{\partial T} f(z)dz = 0$$

for any closed triangle T. Therefore, by Theorem 4.4, there exist a function $F(z)$ such that

$$F'(z) = f(z).$$

But if f has a primitive, then by Theorem 3.4,

$$\int_C f(z)dz$$

depends on the starting point and end point of C. But since C is a closed curve, these points are the same. Hence,

$$\int_C f(z)dz = 0,$$

and this completes the proof of the Theorem. □

EXAMPLE 4.2 Let $C := \{z \in \mathbf{C} : |z| = 1\}$. Show that

$$\int_C \frac{z^2}{z-3}dz = 0.$$

Solution

In the region $|z| \leq 1$, the function $\dfrac{z^2}{z-3}$ is analytic. Hence, by the Cauchy Theorem,

$$\int_C \frac{z^2}{z-3}dz = 0.$$

Remark 4.2 We summarize what we have done in order to prove Theorem 4.5. We first show that if f is analytic on a region D then the integral of f over any boundary of a triangle in D is 0. We use this fact to construct a primitive of f for function analytic on a star-shaped region. We now observe that if $S = \{z| -\pi < \arg z < \pi\}$ then S is a star shaped region with star center 1 (In fact, we can choose any positive real number to be the star center.) Hence if we let

$$\mathrm{Ln}\, z = \int_1^z \frac{1}{\zeta}d\zeta,$$

we find that

$$\frac{d\mathrm{Ln\ z}}{dz} = \frac{1}{z}.$$

Note that when z is real and positive,

$$\int_1^x \frac{1}{t}dt = \ln x.$$

It can be shown that

$$e^{\mathrm{Ln} z} = z$$

and hence $\mathrm{Ln} z$ is an inverse function for e^z. We use the word "an" because $\mathrm{Ln} z + 2\pi ik$ is also an inverse function for e^z for any integer k.

4.3 Extended Cauchy-Goursat Theorem

In this section, we prove a slightly more general result than the Cauchy-Goursat Theorem.

THEOREM 4.6 Let f be continuous on star shaped region and analytic on $S - \{z_0\}$. Then f has a primitive on S and consequently,

$$\int_C f(z)dz = 0$$

for every simple closed curve in S traversed in the counterclockwise direction.

The proof of the above is exactly the same as Theorem 4.5. The only difference is that we need a different version of Lemma 4.3 which we now state.

LEMMA 4.7 Let f be continuous on S and analytic on $S - \{z_0\}$. If T is any triangle contained in S, then

$$\int_{\partial T} f(z)dz = 0.$$

Proof
We split our proof into several cases.

Case 1. If the closed triangle T does not contain z_0, then the conclusion follows using Lemma 4.3.

Case 2. Suppose z_0 is a vertex of T and suppose that

$$|I| = \left|\int_{\partial T} f(z)dz\right| > 0$$

Since $z_0 \in S$ and S is open, we can find $\delta_1 > 0$ such that $B(z_0; \delta_1) \subset S$. Next, f is continuous at z_0 implies that there exists $\delta_2 > 0$ such that

$$|f(z) - f(z_0)| < 1 \quad \text{whenever} \quad |z - z_0| < \delta_2.$$

In other words,

$$|f(z)| \leq 1 + |f(z_0)| \quad \text{whenever} \quad |z - z_0| < \delta_2. \tag{4.11}$$

Next, let

$$\delta_3 = \frac{|I|}{8(|f(z_0)| + 1)}$$

and set $\delta = \min(\delta_1, \delta_2, \delta_3)$.

Choose a and b on the triangle T such that the triangle T_1 formed by z_0, a and b lies inside $B(z_0;\delta)$. Note that

$$\int_{\partial T} f(z)dz = \int_{\partial T_1} f(z)dz.$$

Now, the length of T_1 is less than 4δ (the lengths of $[z_0, a]$ and $[z_0, b]$ are each less than δ and the length of $[a, b]$ is less than 2δ, the diameter of $C(z_0;\delta)$). By (4.11), $|f(z)|$ is bounded by $|f(z_0)| + 1$. Hence, by the ML-formula, we find that

$$\begin{aligned} 0 < |I| &= \left| \int_{\partial T_1} f(z)dz \right| \\ &< (|f(z_0)| + 1)\, 4\delta \\ &< \frac{|I|}{2}, \end{aligned}$$

where we have used the bound $\delta \leq \delta_3$ in the last inequality. Hence

$$\frac{|I|}{2} > |I|$$

and we have a contradiction. This implies that $|I| = 0$.

Case 3. If z_0 lies on the edge of the triangle, we just divide the triangle into two triangles having z_0 as a vertex and apply Case 2.

Case 4. If z_0 lies in the interior of T, we join z_0 to the three vertices to form three triangles with vertex z_0 and apply Case 2.

This completes the proof of the lemma. □

4.4 The Cauchy Integral Formula

We now apply Theorem 4.6 to obtain an important result.

THEOREM 4.8 Let f be analytic in a starshaped region S and let C be a simple closed contour in S traversed in the counterclockwise direction. If w is any point interior to C, then

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w}\, dz.$$

This is called the Cauchy Integral Formula. It tells us that if a function f is analytic within and on a simple closed curve, then the values of f interior to C

are completely determined by the values of f on C. In application, we choose C to be a circle centered at w with radius r such that $C(w;r) \in S$.

Sketch of the proof of Theorem 4.8
Define that function

$$g(z) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{when } z \neq w \\ f'(w) & z = w. \end{cases} .$$

Note that $g(z)$ is continuous in S and analytic at $S - \{w\}$. If C is a simple closed curve in S, then by Theorem 4.6,

$$\int_C g(z)\,dz = 0.$$

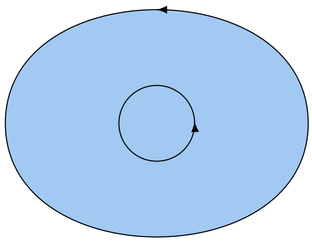
But

$$0 = \int_C g(z)\,dz = \int_C \frac{f(z)-f(w)}{z-w}\,dz = \int_C \frac{f(z)}{z-w}\,dz - \int_C \frac{f(w)}{z-w}\,dz. \tag{4.12}$$

But if C is a simple closed curve containing w then

$$\int_C \frac{1}{z-w}\,dz = \int_{C(w;r)} \frac{1}{z-w}\,dz, \tag{4.13}$$

where $C(w;r)$ is the circle centered at w with radius r traversed in the counterclockwise direction.



By parametrizing $C(w;r)$ using $z(t) = w + re^{it}, 0 \le t \le 2\pi$, we find that

$$\int_{C(w;r)} \frac{1}{z-w}\,dz = 2\pi i.$$

Therefore,

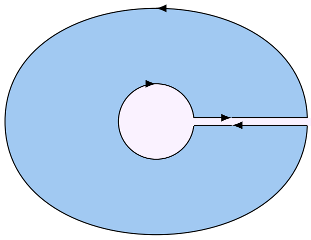
$$\int_C \frac{1}{z-w}\,dz = 2\pi i, \tag{4.14}$$

for any simple closed curve containing w. Hence, we may rewrite (4.12) as

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz,$$

which completes the proof of the theorem. □

Remark 4.3 We have assumed (4.13) for arbitrary closed simple curve C. Note that the region bounded by the following curve



is not star shaped. One can cover this compact region by finitely many open balls with centers in the region. Subdivide the curves so that each subdivision is in an open ball. Since an open ball is star shaped the integral over the contour lying in the ball is 0. By adding these integrals, we obtain (4.13).

EXAMPLE 4.3 Let C be a positively oriented circle $|z| = 2$. Show that

$$\int_C \frac{z}{(9-z^2)(z+i)} dz = \frac{\pi}{5}.$$

Solution

Since the function

$$f(z) = \frac{z}{9-z^2}$$

is analytic within and on C and since the point $z_0 = -i$ is interior to C, the Cauchy integral formula gives

$$\int_C \frac{z}{(9-z^2)(z+i)} dz = \int_C \frac{z/(9-z^2)}{z-(-i)} dz = 2\pi i \left(\frac{-i}{10}\right) = \frac{\pi}{5}.$$

Cauchy's integral formula can also be used to evaluate certain definite integral. We illustrate this in the following examples.

EXAMPLE 4.4 Show that

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta} d\theta = \frac{2\pi}{3}.$$

Solution

Let $\zeta = e^{i\theta}$. Then

$$5 + 4\cos\theta = \frac{1}{\zeta}\left(5\zeta + 2\zeta^2 + 2\right).$$

We deduce that

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta}\,d\theta = \frac{1}{i}\int_{C(0;1)} \frac{1}{2\zeta^2+5\zeta+2}\,d\zeta = \frac{1}{i}\int_{C(0;1)} \frac{1}{2(\zeta+1/2)(\zeta+2)}\,d\zeta.$$

Now, the curve $C(0;1)$ encloses only $-1/2$ and therefore, by Cauchy's integral formula, we deduce that

$$\frac{1}{i}\int_{C(0;1)} \frac{1}{2(\zeta+1/2)(\zeta+2)}\,d\zeta = \frac{1}{i}2\pi i \frac{1}{2(\zeta+2)}\bigg|_{\zeta=-1/2} = \frac{2\pi}{3}$$

and this completes the proof of the identity.

4.5 Liouville's Theorem and the Fundamental Theorem of Algebra

THEOREM 4.9 (Liouville's Theorem) Let $f(z)$ be an entire function. If $f(z)$ is bounded, then f is a constant.

Proof
Suppose for all $z \in \mathbf{C}$, $|f(z)| \le M$, where M is some positive real number. Let a, b be arbitrary distinct complex numbers. Our aim is to show that $f(a) = f(b)$, which will imply that $f(z)$ is a constant function.

Let $R > 0$ be large enough so that $C(0;R)$ encloses a and b. Let C be the contour $C(0;R)$ traversed in the counterclockwise direction. By Cauchy's integral formula, we deduce that

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - a}\,d\zeta$$

and

$$f(b) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - b}\,d\zeta.$$

This implies that

$$f(b) - f(a) = \frac{1}{2\pi i}\int_C f(\zeta)\left(\frac{1}{\zeta-b} - \frac{1}{\zeta-a}\right)d\zeta = \frac{1}{2\pi i}\int_C f(\zeta)\frac{b-a}{(\zeta-b)(\zeta-a)}\,d\zeta.$$

By $ML-$formula, we deduce that

$$|f(b)-f(a)| \le \frac{1}{2\pi}|b-a|\frac{2\pi RM}{(R-|b|)(R-|a|)}.$$

As $R \to \infty$, we deduce that $|f(b) - f(a)| = 0$, or $f(a) = f(b)$. □

EXAMPLE 4.5 Let f be an entire function and suppose that $\operatorname{Re} f(z) < M$ for all $z \in \mathbf{C}$. Prove that f is a constant function.

Solution
Let $g(z) = e^{f(z)}$. Then

$$\left|e^{f(z)}\right| = e^{\operatorname{Re} f(z)} \le e^M.$$

This implies, using Liouville Theorem, that $e^{f(z)}$ is a constant function since $e^{f(z)}$ is entire. Let $f(z) = u + iv$. Then $e^{f(z)} = e^u \cdot e^{iv}$. The fact that $e^{f(z)} = C$ implies that

$$e^u = |C|.$$

Hence,

$$u = \ln |C|.$$

Since the real part of $f(z)$ is a constant, we conclude that $f(z)$ is a constant.

An application of this Theorem is that it yields a simple proof of the Fundamental Theorem of Algebra. This is surprising since we are now using results in analysis to prove results in Algebra. This further shows that topics in mathematics are inter-related.

Before we proceed with the proof of the next theorem, we quote a fact about continuous functions and subsets of $\mathbf{C}$ which are closed and bounded.

THEOREM 4.10 Let f be a continuous function on a region D and S be a closed and bounded set in D. Then $f(S)$ is also closed and bounded.

Remark 4.4 We will use Theorem 4.10 several times in this course to simplify our proofs of various theorems. The proof of the theorem is indirect. We first show that a set in $\mathbf{C}$ is closed and bounded if and only if it is compact (see Appendix for its definition). We then prove that continuous functions send a compact set to a compact set. This then implies that a continuous function sends a closed and bounded set to a closed and bounded set.

THEOREM 4.11 (Fundamental Theorem of Algebra) Every non-constant polynomial with complex coefficients has a zero in $\mathbf{C}$.

Proof
Let $P(z)$ be any non-constant polynomial. Suppose $P(z) \neq 0$ for all $z \in \mathbf{C}$. This implies that $f(z) = \dfrac{1}{P(z)}$ is defined for all $z \in \mathbf{C}$ since $f(z)$ is bounded in $\mathbf{C}$. The function $f(z)$ is entire because its derivative is given by

$$-\frac{P'(z)}{P(z)^2}.$$

Furthermore, if $P(z)$ is non-constant, $P \to \infty$ as $z \to \infty$ and so f is bounded. This is because since $f \to 0$ as $z \to \infty$, for $\epsilon = 1$ there exists M such that

$$|f| \leq 1 \quad \text{for all } |z| > M.$$

For $|z| \leq M$, we see that the set $S = \{z||z| \leq M\}$ is closed and bounded. By Theorem 4.10, we know that $f(S)$ is bounded, say,

$$|f(z)| \leq B$$

for all $|z| \leq M$. Hence for all $z \in \mathbf{C}$,

$$|f| \leq \max(B, 1)$$

and f is a bounded entire function.

By Liouville's Theorem, f must be a constant and so, $P(z)$ must be a constant. This contradicts our choice of $P(z)$. □

By induction, we see that any polynomial with complex coefficients must factor into linear factors, in other words,

$$P(z) = a_n z^n + \cdots + a_1 z + a_0 = a_n(z - \alpha_1) \cdots (z - \alpha_n).$$

In Chapter 1, we indicate that i may be defined as the root of $z^2 + 1 = 0$. A natural question to ask is if we consider all possible polynomials with coefficients in $\mathbf{C}$, say for example, $z^2 + i = 0$, will we discover new numbers analogous to that of i? Or do we have to define new numbers to solve these polynomials? The above observation says that we do not have to define more numbers. In fact, all polynomials factor into linear factors in $\mathbf{C}$.

4.6 Appendix : Compact sets in $\mathbf{C}$

DEFINITION 4.4 A set $S \subset \mathbf{C}$ is compact if it satisfies the Heine-Borel property, namely, or every open covering $\mathcal{C}$ of S, there exists a finite subcovering of S in $\mathcal{C}$.

This means that if $S \subset \bigcup_\alpha U_\alpha$ with $U_\alpha \in \mathcal{C}$, then

$$S \subset \bigcup_{j=1}^{k} U_{\alpha_j}.$$

It can be shown that if S is a subset of $\mathcal{C}$ then S is a compact if and only if S is closed and bounded.

Let f be a function from a region X to Y. A function is continuous on a region

X if for every set V that is "open in Y",[1] the set $f^{-1}(V)$ is "open in X". We can show that our old notion of continuous function satisfies the above property (see Appendix of Chapter 2).

We now sketch the proof of Theorem 4.10.

Sketch of the proof of Theorem 4.10
Let $\mathcal{C}$ be a covering of $f(S)$. Then

$$f(S) \subset \bigcup_{\alpha} \mathcal{O}_{\alpha},$$

where $\mathcal{O}_{\alpha}$ is open in $\mathbf{C}$. Let $U_{\alpha} = \mathcal{O}_{\alpha} \cap Y$. Then $f(S) \subset \bigcup_{\alpha} U_{\alpha}$. This means that

$$S \subset f^{-1}\left(\bigcup_{\alpha} U_{\alpha}\right) = \bigcup_{\alpha} f^{-1}\left(U_{\alpha}\right).$$

Note that because f is continuous, $f^{-1}(U_{\alpha})$ is "open in X". Let $f^{-1}(U_{\alpha}) = \mathcal{O}^{*}_{\alpha} \cap X$, with $\mathcal{O}^{*}_{\alpha}$ open set in $\mathcal{C}$. Since S is compact,

$$S \subset \bigcup_{j=1}^{k} \mathcal{O}^{*}_{\alpha_j}.$$

Hence,

$$S \cap X = S \subset \left(\bigcup_{j=1}^{k} \mathcal{O}^{*}_{\alpha_j}\right) \cap X = \bigcup_{j=1}^{k} f^{-1}\left(U_{\alpha_j}\right).$$

Hence

$$f(S) \subset \bigcup_{j=1}^{k} \left(\mathcal{O}_{\alpha_j}\right),$$

and therefore $f(S)$ is compact. □

Our aim is conclude that if S is compact and f is continuous then $|f(S)|$ is bounded since a set in $\mathbf{C}$ is compact if and only if it is closed and bounded.

We will next prove that a set S in $\mathbf{C}$ is compact if and only if S is closed and bounded.

We do this in several steps.

LEMMA 4.12 If $S \subset \mathbf{C}$ is compact, then S is bounded.

Proof
It is clear that

$$S \subset \bigcup_{s \in S} B(s; 1).$$

[1] We say that U is open in S if $U = O \cap S$ for some open set O in $\mathbf{C}$.

Since S is compact, it is covered by finitely many such open balls and we have

$$S \subset \bigcup_{j=1}^{k} B(s_j; 1).$$

Let $d = \max_{1 \le i < j \le k} |s_i - s_j|$. Given any $u, v \in S$, $u \in B(s_m; 1)$ and $v \in B(s_n; 1)$ for some integers m and n between 1 and k. Now, $|u - v| \le |u - s_m| + |s_m - s_n| + |v - s_n| < 2 + d$. Hence the diameter of S is finite and S is bounded by $B(0; |u| + \text{diam} S)$. □

LEMMA 4.13 If $S \subset \mathbf{C}$ is compact, then S is closed.

Proof
To show that S is closed, we show that $S = \overline{S}$. Suppose $s \in \partial S$ and $s \notin S$. Then $B(s; \epsilon) \cap S \neq \phi$ and $B(s; \epsilon) \cap S^c \neq \phi$ for all $\epsilon > 0$. In particular, for $n \in \mathbf{Z}^+$,

$$B\left(s, \frac{1}{n}\right) \cap S \neq \phi.$$

This implies that

$$\overline{B\left(s, \frac{1}{n}\right)} \cap S \neq \phi.$$

Let $G_n = \left(\overline{B\left(s, \frac{1}{n}\right)}\right)^c$. Note that G_n is open. Now,

$$\bigcup \left(\overline{B\left(s, \frac{1}{n}\right)}\right)^c = \left(\bigcap \overline{B\left(s, \frac{1}{n}\right)}\right)^c.$$

But $\{\overline{B\left(s, \frac{1}{n}\right)} | n \in \mathbf{Z}^+\}$ is a collection of nested closed sets and by Cantor's Theorem (Lemma 4.2),

$$\bigcap \overline{B\left(s, \frac{1}{n}\right)} = \{s\}.$$

Therefore $\bigcup G_n = (\{s\})^c$ covers S ($s \notin S$). Since S is compact, S is covered by finitely many sets G_{n_j} or

$$S \subset \bigcup_{j=1}^{k} G_{n_j}.$$

This means that

$$S \cap \left(\bigcup_{j=1}^{k} G_{n_j}\right)^c = \phi,$$

with $n_1 < n_2 < \cdots < n_k$, or

$$S \cap \overline{B\left(s, \frac{1}{n_1}\right)} \cap \overline{B\left(s, \frac{1}{n_2}\right)} \cdots \cap \overline{B\left(s, \frac{1}{n_k}\right)} = S \cap \overline{B\left(s, \frac{1}{n_k}\right)} = \phi.$$

This contradicts the fact that $s \in \partial S$. Therefore S is closed. □

We are now left with the proof that if S is closed and bounded then S is compact [2]

DEFINITION 4.5 We say that a set S is totally bounded if for every $\epsilon > 0$, S can be covered by finitely many open balls of radius ϵ.

LEMMA 4.14 If S is bounded then it is totally bounded.

Proof
If the set is bounded and $\epsilon > 0$ is given, we can divide the set S using squares of side $\epsilon/\sqrt{2}$. Then the finite number of balls with vertices of the squares as centers would cover S. So, S is totally bounded. □

LEMMA 4.15 If S is closed subset of a complete set, S is complete.

Proof
Let $\{s_k\}$ be a sequence of numbers from S that is Cauchy. Since $\mathbf{C}$ is complete, $s_k \to a$ for some $a \in \mathbf{C}$. We claim that $a \in S$. If $a \notin S$, $a \in S^c$ and $B(a;\epsilon) \subset S^c$ for some $\epsilon > 0$ since S^c is open. This means that $B(a;\epsilon) \cap S = \phi$ and $|s_k - a| \geq \epsilon$. But this contradicts the fact that $s_k \to a$. Hence S is complete. □

LEMMA 4.16 If S is closed and bounded then S is compact.

Proof
From the previous two lemmas, we can assume that S is complete and totally bounded. Suppose $\mathcal{C} = \{\mathcal{O}_j\}$ is an open covering for S without any finite sub-covering.

Let $\epsilon_1 = \frac{1}{2}$. The set S is totally bounded implies that S can be covered by finitely many open balls of radius ϵ_1. Write

$$S = \bigcup_{\ell=1}^{m_1} (B(a_\ell;\epsilon_1) \cap S)$$

where we only include those open balls with non-empty intersection with S.

By assumption, S cannot be covered by finitely many sets $\mathcal{O}_j$'s in $\mathcal{C}$. Hence, there is an open ball $B(x_1;\epsilon_1)$ centered at x_1 with radius ϵ_1 such that then non-empty set $B(x_1;\epsilon_1) \cap S$ cannot be covered by finitely many open sets $\mathcal{O}_j$'s

[2] This is not true in general for complete metric space. It is true for $\mathbf{C}$ and $\mathbf{R}$.

in $\mathcal{C}$. Now, $B(x_1;\epsilon_1)\cap S$ is totally bounded since it is a subset of a totally bounded set S. Therefore $B(x_1;\epsilon_1)$ can be covered by finitely many open balls of radius $\epsilon_2 = 2^{-2}$. Once again, among these open balls, there exists an open ball $B(x_2;\epsilon_2)$ such that the non-empty set $B(x_2;\epsilon_2)\cap S$ cannot be covered by finitely many $\mathcal{O}_j$'s in $\mathcal{C}$. Continuing with this construction, we obtain a sequence of sets $\{B(x_k;\epsilon_k)\cap S\}$ that cannot be covered by finitely many sets in $\mathcal{C}$. For each k, let $s_k \in (B(x_k;\epsilon_k)\cap S)$. Note that

$$\begin{aligned}|s_n - s_{n+1}| &\le |s_n - x_n| + |x_n - x_{n+1}| + |x_{n+1} - s_{n+1}|\\ &< \frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} \le \frac{3}{2^n}.\end{aligned}$$

Hence,

$$|s_m - s_{m+p}| \le |s_m - s_{m+1}| + \cdots + |s_{m+p-1} - s_{m+p}| < \frac{6}{2^m}$$

Therefore, the sequence $\{s_k\}$ is a Cauchy sequence. Since S is complete, $s_k \to a$ with $a \in S$. Now S is covered by $\mathcal{O}_j$'s in $\mathcal{C}$ and therefore, $a \in \mathcal{O}$ for some open set $\mathcal{O}$ in $\mathcal{C}$. This implies that $B(a;\delta) \subset \mathcal{O}$ for some $\delta > 0$ since $\mathcal{O}$ is open. Now, since $s_k \to a$, there exists M such that

$$|s_k - a| < \frac{\delta}{3}$$

for all $k > M$. We choose $k > M$, say $k = K$, such that

$$\epsilon_K = 2^{-K} < \frac{\delta}{3}.$$

Now, consider the set $B(x_K;\epsilon_K)\cap S$. By assumption,

$$s_K \in B(x_K;\epsilon_K)\cap S.$$

Let $b \in B(x_K;\epsilon_K)\cap S$. Then

$$\begin{aligned}|b-a| &= |b - x_K + x_K - s_K + s_K - a|\\ &\le |b - x_K| + |x_K - s_K| + |s_K - a|\\ &\le \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.\end{aligned}$$

Therefore, $b \in B(a;\delta) \subset \mathcal{O}$. Hence, $(B(x_K;\epsilon_K)\cap S) \subset \mathcal{O}$ and is therefore covered by a finite number of sets in $\mathcal{C}$. This contradicts our choice of $B(x_K;\epsilon_K)$ and so S is covered by finitely many sets in $\mathcal{C}$. □

Remark 4.5 In the proof of the above Lemma, one can start with open balls with centers in S. For suppose that $B(x;\epsilon/2)$ is one of the finitely many balls of radius $\epsilon/2$ covering S such that $B(x;\epsilon/2)\cap S \ne \phi$ and that $x \notin S$. Then let $s \in B(x;\epsilon/2)$. The ball $B(s;\epsilon)$ covers $B(x;\epsilon/2)\cap S$ and hence, we obtain an open covering of S by finitely by open balls $B(s;\epsilon)$ with $s \in S$. Using these balls with centers in S, we obtain another proof of the Lemma. For more details, see p. 61 of Ahlfors.

5 Cauchy's Integral formulas and their applications

5.1 First and Second derivative of Analytic Functions

We will prove in this section that if a function is analytic at a point, its derivatives of all orders exist at that point and are themselves analytic there. In other words, if $f(z)$ is analytic at $z = w$ then $f^{(n)}(z)$ is analytic at $z = w$ for $n \geq 1$.

THEOREM 5.1 Let f be analytic on a starshaped region and suppose C is a simple closed contour in S traversed in the counterclockwise direction. If w is any point interior to C, then

$$f'(w) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - w)^2} d\zeta.$$

Proof
First, recall the Cauchy Integral formula

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - w} \, d\zeta.$$

Choose $\delta > 0$ such that $B(w;\delta)$ is contained in the region enclosed by C. Let $z \in B(w;\delta)$. Then

$$\begin{aligned}
&\frac{f(z) - f(w)}{z - w} - \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - w)^2} d\zeta \\
&\quad = \frac{1}{2\pi i} \left(\frac{1}{z - w} \int_C f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} - \frac{z - w}{(\zeta - w)^2} \right) d\zeta \right) \\
&\quad = \frac{1}{2\pi i} \int_C f(\zeta) \frac{(z - w)}{(\zeta - z)(\zeta - w)^2} d\zeta.
\end{aligned}$$

Recall that $\delta > 0$ is chosen so that $B(w;\delta)$ is contained in the region enclosed by C and $\overline{B(w;\delta)} \cap C = \phi$. Let d be the minimum distance from $\overline{B(w;\delta)}$ to C. [1] Then

$$|(\zeta - z)(\zeta - w)^2| \geq d^3.$$

[1] The number d exists because for $v \in C$, the function $f(v) = \min_{a \in \overline{B(w;\delta)}}(|v - a|)$ is a continuous function in v and and is greater than 0 because $\overline{B(w;\delta)} \cap C = \phi$.

Since f is continuous, by Theorem 4.10, $|f(\zeta)| < M$ for some $M > 0$. Therefore, we find that

$$\left|\frac{f(\zeta)}{(\zeta - z)(\zeta - w)^2}\right| \leq \frac{M}{d^3}.$$

Denoting the length of C by L and using ML-formula, we deduce that

$$\left|\frac{f(z) - f(w)}{z - w} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - w)^2}d\zeta\right| < \frac{1}{2\pi}\frac{LM}{d^3}|z - w| < \epsilon$$

whenever $|z - w| < \min(\delta, 2\pi\epsilon d^3/(LM))$. This concludes the proof of the theorem. □

THEOREM 5.2 Let f be analytic on a starshaped region and suppose C is a simple closed contour in S traversed in the counterclockwise direction. If w is any point interior to C, then

$$f''(w) = \frac{2}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - w)^3}d\zeta.$$

Proof
Choose $\delta > 0$ such that $B(w;\delta)$ is contained in the region enclosed by C. Let $z \in B(w;\delta)$. Then

$$\begin{aligned}
&\frac{f'(z) - f'(w)}{z - w} - \frac{2}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - w)^3}d\zeta \\
&\quad = \frac{1}{2\pi i(z - w)}\int_C f(\zeta)\left(\frac{1}{(\zeta - z)^2} - \frac{1}{(\zeta - w)^2} - \frac{2(z - w)}{(\zeta - w)^3}\right)d\zeta \\
&\quad = \frac{1}{2\pi i}\int_C f(\zeta)\frac{(2\zeta - w - z)(\zeta - w) - 2(\zeta - z)^2}{(\zeta - z)^2(\zeta - w)^3}d\zeta \\
= &\quad = \frac{1}{2\pi i}\int_C f(\zeta)\frac{(z - w)(3\zeta - w - 2z)}{(\zeta - z)^2(\zeta - w)^3}d\zeta,
\end{aligned}$$

where the last equality holds since

$$\begin{aligned}
(2\zeta - w - z)(\zeta - w) - 2(\zeta - z)^2 &= (\zeta - w + \zeta - z)(\zeta - z + z - w) - 2(\zeta - z)^2 \\
&= (\zeta - z)(\zeta - w) - (\zeta - z)^2 + (z - w)(\zeta - w + \zeta - z) \\
&= (\zeta - z)(\zeta - w - \zeta + w) + (z - w)(2\zeta - w - z) \\
&= (z - w)(3\zeta - 2z - w).
\end{aligned}$$

Now, as in the proof of the previous theorem, we find that by Theorem 4.10, $|f(\zeta)| \leq M$ for all $\zeta \in C$. Furthermore, if d is the minimum distance from $\overline{B(w;\delta)}$ to C, then

$$|(\zeta - z)^2(\zeta - w)^3| \geq d^5$$

and hence,

$$\left| f(\zeta) \frac{3\zeta - w - 2z}{(\zeta - z)^2(\zeta - w)^3} \right| \leq \frac{M}{d^5}$$

for all ζ on the contour C. Denoting the length of C by L and using ML-formula, we deduce that

$$\left| \frac{f'(z) - f'(w)}{z - w} - \frac{2}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - w)^3} d\zeta \right| < \frac{1}{2\pi} LM|z - w|/d^5 < \epsilon$$

whenever $|z - w| < \min(\delta, 2\pi\epsilon d^5/(LM))$. This concludes the proof of the theorem. □

From Theorem 5.1 and Theorem 5.2, we deduce the following important result which we have assumed when we show that the real and imaginary parts of an analytic function is harmonic.

COROLLARY 5.3 If f is analytic on a region D then f' is analytic on D.

By applying Corollary 5.3 repeatedly, we deduce the following corollary.

COROLLARY 5.4 If f is analytic on a region D, then the k-th derivative $f^{(k)}$ is analytic on D for all positive integers k.

5.2 Higher derivatives of analytic functions

THEOREM 5.5 Let f be analytic on a starshaped region and suppose C is a simple closed contour in S traversed in the counterclockwise direction that encloses w. Then

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - w)^{n+1}} dz.$$

Proof

We proceed by using mathematical induction. Note that the case for $k = 1$ is Theorem 5.1. Suppose the result is true for $k = n - 1$. By Corollary 5.3, we note that we may apply Cauchy's integral formula for $(n-1)$-th derivative to f' since f' is analytic in D whenever f is analytic in D. Hence,

$$(f')^{(n-1)}(w) = \frac{(n-1)!}{2\pi i} \int_C \frac{f'(\zeta)}{(\zeta - w)^n} dz. \tag{5.1}$$

On the other hand, we know that

$$\left(\frac{f(\zeta)}{(\zeta-w)^n}\right)' = \frac{f'(\zeta)}{(\zeta-w)^n} - n\frac{f(\zeta)}{(\zeta-w)^{n+1}}$$

and hence by the analogue of the Fundamental Theorem of Calculus, we find that

$$\frac{1}{2\pi i}\int_C \frac{f'(\zeta)}{(\zeta-w)^n}\,d\zeta = \frac{n}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-w)^{n+1}}\,d\zeta \tag{5.2}$$

since C is a simple closed curve. Combining (5.1) and (5.2), we complete the proof of the theorem. □

EXAMPLE 5.1 Evaluate the integral

$$\int_C \frac{5z^2+2z+1}{(z-i)^3}dz$$

where C is the circle with center 0 and radius 2.

Solution

By Cauchy's integral formula for $f''(z)$, we conclude that

$$\int_C \frac{5z^2+2z+1}{(z-i)^3}dz = \pi i f''(i) = 10\pi i,$$

since $f''(i) = 10$.

EXAMPLE 5.2 When $f(z) = 1$,

$$\int_C \frac{1}{z-z_0}\,dz = 2\pi i,$$

and

$$\int_C \frac{1}{(z-z_0)^k}\,dz = 0$$

for $k \geq 2$.

5.3 Cauchy's Integral formula and Extended Liouville Theorem

The Liouville Theorem (see Theorem 4.9) states that if $f(z)$ is a bounded entire function, then $f(z)$ is a constant. We now give the following extended version of Liouville's Theorem:

THEOREM 5.6 If f is entire and if, for some integer $k \geq 0$, there exist positive constants A and B such that

$$|f(z)| \leq A + B|z|^k,$$

then f is a polynomial of degree at most k.

Proof
We first note that if $f^{(m+1)}(z) = 0$ for all $z \in \mathbf{C}$, then $f(z)$ is a polynomial of degree at most m. We can prove this claim by induction on m. Suppose $j = 0$, then this means that $f'(z) = 0$ and we have seen (using Cauchy-Riemann equations) that $f(z)$ must be a constant. Suppose the claim is true for $j = m-1$. Now $f^{(m+1)}(z) = 0$ implies that $(f')^{(m)} = 0$. By induction hypothesis, this implies that $f'(z)$ is a polynomial of degree at most $m-1$. Now this polynomial has a primitive (obtained by integrating the polynomial $f'(z)$) $f(z)$ which is a polynomial of degree at most m and this proves our claim.

To establish the extended Liouville Theorem, it suffices to show that $f^{(m+1)}(z) = 0$ for all $z \in \mathbf{C}$. Let $R > 0$ be a positive real number. Let $w \in \mathbf{C}$. By Cauchy's integral formula,

$$f^{(k+1)}(w) = \frac{k!}{2\pi i} \int_{C(0;R)} \frac{f(\zeta)}{(\zeta - w)^{k+2}} d\zeta.$$

By ML-formula and the bound given for $f(z)$, we deduce that

$$|f^{(k+1)}(w)| \leq \frac{k!}{2\pi} \frac{R(A + BR^k)}{(R - |w|)^{k+2}}.$$

The right hand side tends to 0 as R tends to ∞. Hence,

$$|f^{(k+1)}(w)| = 0$$

or $f^{(k+1)}(w) = 0$. Since w is arbitrarily chosen, we conclude that $f^{(k+1)}(z) = 0$ for all $z \in \mathbf{C}$. □

EXAMPLE 5.3 Suppose that $f(z)$ is a non-constant entire and for all $z \in \mathbf{C}$,

$$|f(z)| \leq A + B|z|^{3/2}$$

for all positive real numbers A and B. Show that $f(z)$ is a linear polynomial.

Solution
For $|z| \leq 1$, we find that $|f(z)| \leq C$ for some $C > 0$ since $f(z)$ is continuous and its image on a bounded set is a bounded set. For $|z| > 1$, then $|z|^{3/2} \leq |z|^2$ and

$$|f(z)| \leq A + B|z|^{3/2} < A' + B|z|^2,$$

where $A' > C$. Using Extended Liouville Theorem, we conclude that $f(z) = a_2 z^2 + a_1 z + a_0$. If $a_2 = 0$, then $f(z)$ is a linear polynomial. Suppose $a_2 \neq 0$. Then

$$|a_2||z|^2 - |a_1||z| - |a_0| \leq |f(z)| \leq A + B|z|^{3/2}$$

implies that for sufficiently large R (take $R > \max(1, \sqrt{|a_1| + |a_0|})$) and $|z| = R$,

$$|a_2|R^{1/2} - |a_1|/R^{1/2} - |a_0|/R^{3/2} \leq A/R^{3/2} + B.$$

The left hand side tends to ∞ as R tends to ∞ and the right hand side is bounded. This is impossible and so, $a_2 = 0$.

5.4 Morera's Theorem and Extended Liouville Theorem

We will now use Corollary 5.3 to prove an important result known as Morera's Theorem.

THEOREM 5.7 (Morera's Theorem) Let f be a continuous function on a star-shaped region D. Let T be a closed triangle in D and ∂T be the boundary of T traversed in the counterclockwise direction. If

$$\int_{\partial T} f(z)dz = 0$$

for all $T \subset D$, then f is analytic in D.

Proof
We first note that this is very similar to the statement when we wish to derive primitives for f in our proof of the Cauchy Goursat Theorem. However, we did not conclude that f is analytic.

Let $z_0 \in D$. Since D is a region, there exists $\epsilon > 0$ such that $B(z_0; \epsilon) \subset D$. Now, a open ball is a star shaped domain and the conditions guaranteed that we could construct a primitive for f, namely,

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta.$$

Now, $F'(z) = f(z)$ and so, F is analytic on z_0. But by Corollary 5.3, we conclude that F' is analytic at z_0. This means that f is analytic at z_0. □

We now use Morera's Theorem to prove the following result.

THEOREM 5.8 Let f be an entire function and

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a. \end{cases}.$$

Then g is entire.

Proof
From the definition of g we see that g is analytic at $\mathbf{C} - \{a\}$ and continuous at $z = a$. Let T be any triangle in $\mathbf{C}$. By the extended Cauchy Goursat Theorem (Theorem 4.6), we conclude that

$$\int_{\partial T} g(\zeta) d\zeta = 0.$$

Hence, by Morera's Theorem (Theorem 5.7), we conclude that g is analytic at a and hence g is entire. □

We can now give another proof of extended Liouville Theorem.

EXAMPLE 5.4 Give a proof of the extended Liouville Theorem using Morera's Theorem.

Solution
We prove by induction on k. If $k = 1$, then

$$|f(z)| \leq A + B|z|.$$

Define

$$g(z) = \begin{cases} \dfrac{f(z) - f(0)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0. \end{cases}.$$

By Morera's Theorem, $g(z)$ is entire. When $z \neq 0$ and $|z| \leq 1$, $g(z)$ is bounded by C, say. When $|z| > 1$,

$$|g(z)| \leq \frac{A}{|z|} + B \leq A + B.$$

This implies that $g(z)$ is a bounded entire function and hence a constant. Hence

$$f(z) = f(0) + zg(z) = f(0) + Cz$$

is a linear polynomial since $g(z) = C$.

Suppose the claim is true for any entire functions $F(z)$ satisfying

$$|F(z)| \leq C + D|z|^{k-1}.$$

Let $f(z)$ be entire and

$$|f(z)| \leq A + B|z|^k.$$

Define

$$g(z) = \begin{cases} \dfrac{f(z) - f(0)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0. \end{cases} .$$

Note that $g(z)$ is entire by Morera's Theorem. For $|z| \leq 1$, $g(z)$ is bounded, say, by M^*. For $|z| > 1$, we conclude that

$$|g(z)| \leq \frac{A}{|z|} + B|z|^{k-1} \leq A + B|z|^{k-1}.$$

Hence, for all z,

$$|g(z)| \leq A' + B'|z|^{k-1},$$

where $A' > A{+}B$. By induction hypothesis, we conclude that $g(z)$ is a polynomial of degree at most $(k-1)$.

Since

$$g(z) = \frac{f(z) - f(0)}{z}$$

we conclude that $f(z)$ is a polynomial of degree at most k.

EXAMPLE 5.5 Suppose $f(z)$ is entire and $|f'(z)| \leq |z|$ for all $z \in \mathbf{C}$. Show that

$$f(z) = a + bz^2$$

with $|b| \leq 1/2$.

Solution

The function $f'(z)$ is entire since f is entire. We have $|f'(z)| \leq |z|$ implies that $f'(z) = Az + B$ by the extended Liouville theorem. But from the inequality $|f'(z)| \leq |z|$ shows that $f'(0) = 0$ and hence $B = 0$. Now $|f'(z)| = |Az| \leq |z|$ implies that $|A| \leq 1$.

Next, $f' = Az$ implies that $f = Az^2/2 + C$ and $|A/2| \leq \frac{1}{2}$.

5.5 Mean Value Theorem and the Maximum Modulus Theorem

We now examine some local behavior of analytic functions.

THEOREM 5.9 (Mean Value Theorem) If f is analytic in a region D and $\alpha \in D$, then $f(\alpha)$ is equal to the mean value of f taken around the boundary of any ball centered at α and contained in D. That is

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta})d\theta$$

when $B(\alpha; r) \subset D$.

Proof
From Cauchy Integral formula, we have

$$f(\alpha) = \frac{1}{2\pi i} \int_{C(\alpha;r)} \frac{f(\zeta)}{\zeta - \alpha} d\zeta.$$

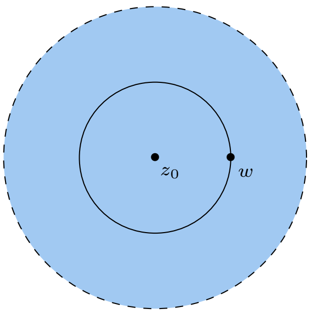
Let $\zeta = re^{i\theta} + \alpha$. Then we find that

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta}) d\theta.$$

□

THEOREM 5.10 (Maximum Modulus Theorem on an open ball) Suppose that $f(z)$ is analytic throughout a neighborhood $|z - z_0| < R$ of a point z_0. If $|f(z)| \leq |f(z_0)|$ for each point z in that neighborhood, then $f(z)$ has the constant value $f(z_0)$ throughout the neighborhood.

Proof
Our aim is to show that if $|f(z)|$ is maximum for some $z = z_0$ in $B(z_0; R)$ then $f(z)$ is a constant on $B(z_0; R)$. Let w be an arbitrary point in $B(z_0; R)$.



Let $r = |w - z_0| < R$. From the Mean Value Theorem (Theorem 5.9), we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

It follows that

$$|f(z_0)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta. \tag{5.3}$$

By assumption that $|f(z_0)|$ is maximum, we have $|f(z_0)| \geq |f(z)|$ for $z \in$

$C(z_0; r)$. Hence,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta \le |f(z_0)|.$$

Together with (5.3), we deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta = |f(z_0)|,$$

or

$$\int_0^{2\pi} |f(z_0)| - |f(z_0 + re^{i\theta})| d\theta = 0 \tag{5.4}$$

We claim that $|f(w)| = |f(z_0)|$ for all $w \in C(z_0; r)$. Suppose not. Then there exist T such that $|f(z_0 + re^{2iT})| < |f(z_0)|, 0 \le T \le 2\pi$. Let $F(t) = |f(z_0)| - |f(z_0 + re^{2it})|$. Then we have $|F(T)| > 0$. Let $|F(T)| = h > 0$. Since $F(t)$ is continuous, for $h/2 > 0$, there exists $\delta > 0$ such that if $|t - T| < \delta$ then

$$|F(t) - F(T)| < \frac{h}{2},$$

or

$$|F(T)| - |F(t)| < |F(T) - F(t)| < \frac{h}{2}.$$

Therefore,

$$|F(t)| > \frac{h}{2}.$$

Hence,

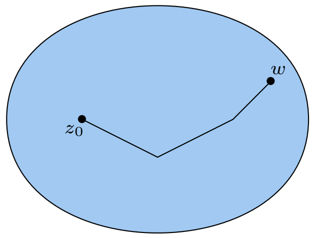
$$\int_{T-\delta}^{T+\delta} |F(t)| dt > \frac{h}{2} 2\delta > 0$$

and this contradicts (5.4). This implies $|f(z_0)| = |f(z_0 + re^{it})|, 0 \le t \le 2\pi$ and in particular, $|f(z_0)| = |f(w)|$. Since w is arbitrary, we conclude that $|f(z_0)| = |f(z)|$ for all $z \in B(z_0; R)$. Now, by Example 2.26, we conclude that f is constant on $B(z_0; R)$. □

THEOREM 5.11 (Maximum modulus principle for a region) If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \le |f(z_0)|$ for all points z in it.

Proof

We wish to prove that for any $w \in D$, $f(w) = f(z_0)$. It suffices to show that $|f(w)| = |f(z_0)|$ for all $w \in D$. Since D is a region, there is a polygonal line from z_0 to w (see the diagram below):



To show that $|f(w)| = |f(z_0)|$, it suffices to show that if $[z_0, z_1]$ is a line, then $|f(z_0)| = |f(z_1)|$. By continuing along the polygonal line and using the result for line segment, we conclude that $|f(w)| = |f(z_0)|$.

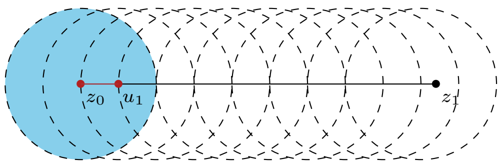
Now, the line segment $[z_0, z_1]$ is a compact set (closed and bounded in $\mathbf{C}$). Since D is open, for each $v \in [z_0, z_1]$, there exists $\delta_v > 0$ such that $B(v; \delta_v) \subset D$. Note that $\{B(v; \delta_v) | v \in [z_0, z_1]\}$ is an open cover for $[z_0, z_1]$. Since $[z_0, z_1]$ is compact, there exists a finite number of u_j such that

$$[z_0, z_1] \subset \bigcup_{j=1}^{K} B(u_j; \delta_{v_j}).$$

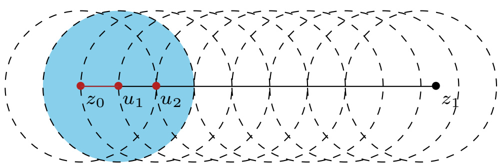
Taking $\epsilon = \min_{1 \le j \le K} \delta_{u_j}$, we conclude that

$$[z_0, z_1] \subset \bigcup_{j=1}^{K} B(u_j; \epsilon).$$

We now have the following situation:



By the maximum modulus principle for open ball, we see that $f(z_0) = f(u_1)$. We then continue the process (see the following diagram where the red denotes those points t with $f(t) = f(z_0)$ and deduce that $f(u_1) = f(u_2)$.



This shows that $f(z_1) = f(z_0)$ if $[z_0, z_1] \subset D$ and by the remark in the beginning of the proof, we conclude that $f(z)$ is constant on D.

□

COROLLARY 5.12 Suppose a function f is continuous in a closed and bounded region D and it is analytic and not constant in the interior of D. Then the maximum value of $|f(z)|$ in D, which is always reached, occurs somewhere on the boundary of D and never in the interior.

Proof
The function f is a continuous function on a closed and bounded set and so, its image is closed and bounded by Theorem 4.10. Let $M = \sup_{z \in D} |f(z)|$. Then there exist $u \in D$ such that $|f(w)| = M$ since $f(D)$ is a closed set (see Remark 5.1). So the maximum modulus is attained. Now since f is not constant, this value cannot be attained in the interior of D, hence the maximum modulus is attained at ∂D.

□

EXAMPLE 5.6 (Bak-Newman, p. 84, Problem 6) Suppose f is a non-constant analytic function in the annulus: $1 \leq |z| \leq 2$, that $|f| \leq 1$ for $|z| = 1$ and that $|f| \leq 4$ for $|z| = 2$. Prove that $|f(z)| \leq |z|^2$ throughout the annulus.

Solution
Let $g(z) = f(z)/z^2$. Then from hypothesis,

$$|g(z)| = \frac{|f(z)|}{|z|^2} \leq 1$$

when $|z| = 1$ and $|z| = 2$. Hence, $|g(z)| \leq 1$ on the boundary of the annulus. Therefore, by maximum modulus principle, $|g(z)| \leq 1$ on $1 \leq |z| \leq 2$. This shows that $|f(z)| \leq |z|^2$ on $1 \leq |z| \leq 2$.

Remark 5.1 Let $M = \sup_{z \in \overline{D}} |f(z)|$ where $\overline{D}$ is the closure of D. Then for $\epsilon = 1/n$, we know that $M - \epsilon$ cannot be a upper bound and hence, there exists z_n such that $M - 1/n < |f(z_n)| < M$. We thus create a sequence $\{|f(z_n)|\}$ that converges to M. Since $|f|$ is continuous, $|f(\overline{D})|$ is complete. Therefore $|f(z_n)|$ converges to $|f(z_0)|$ for some $z_0 \in \overline{D}$ and so, the maximum M is attained by some $z_0 \in \overline{D}$.

6 Series

6.1 Convergence of Sequences and Series

In the proof of Theorem 4.2, we have already seen the definition of a sequence. We needed the notion of Cauchy sequence to define complete space. We now revisit infinite sequence.

DEFINITION 6.1 An infinite sequence of complex numbers

$$z_1, z_2, \cdots, z_n, \cdots,$$

has a limit ℓ if for each positive ϵ, there exists a positive integer N_ϵ such that

$$|z_n - \ell| < \epsilon \quad \text{whenever} \quad n \geq N_\epsilon.$$

Given a sequence $\{z_k\}_{k=1}^{\infty}$. We construct the sequence $\{S_k\}_{k=1}^{\infty}$ where

$$S_k = \sum_{j=1}^{k} z_j.$$

This sequence is called the sequence of partial sums associated with $\{z_k\}_{k=1}^{\infty}$.

If the new sequence $\{S_n\}_{n=1}^{\infty}$ has a limit S, then we say that the infinite series

$$\sum_{k=1}^{\infty} z_k$$

converges and write

$$\sum_{k=1}^{\infty} z_k = S = \lim_{n\to\infty} S_n.$$

The expression

$$\sum_{k=1}^{\infty} z_k$$

is called an infinite series associated with $\{z_k\}_{k=1}^{\infty}$. If the limit of the sequence

$\{S_n\}_{n=1}^{\infty}$ fails to exist, then we say that the infinite series

$$\sum_{k=1}^{\infty} z_k$$

diverges.

6.2 Taylor Series

We turn now to Taylor's Theorem.

THEOREM 6.1 Suppose that a function f is analytic throughout an open ball $B(z_0; R)$. Then at each $z \in B(z_0; R)$, $f(z)$ has the series representation

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad (|z - z_0| < R),$$

where

$$a_k = \frac{f^{(k)}(z_0)}{k!}, k = 0, 1, 2, \cdots .$$

The above theorem is the familiar Taylor series (when restricted to real variables) from Calculus.

Proof
First, set $z_0 = 0$ and suppose f is analytic in $|z| < R$. Let z be chosen and suppose that $|z| = r < R$. Choose R_1 such that $r < R_1 < R$ and let C_{R_1} be the circle $C(0; R_1)$ traversed in the anti-clockwise direction. By the Cauchy Integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{C_{R_1}} \frac{f(s)}{s - z} \, ds.$$

Now,

$$\begin{aligned} \frac{1}{s-z} &= \frac{1}{s}\left(\frac{1}{1-\frac{z}{s}}\right) = \frac{1}{s}\left(1 + \frac{1}{1-\frac{z}{s}} - 1\right) \\ &= \frac{1}{s}\left(1 + \frac{1-1+\frac{z}{s}}{1-\frac{z}{s}}\right) = \frac{1}{s}\left(1 + \frac{\frac{z}{s}}{1-\frac{z}{s}}\right) \end{aligned}$$

$$= \frac{1}{s}\left(1 + \frac{z}{s}\left(\frac{1}{1-\frac{z}{s}}\right)\right) = \frac{1}{s}\left(1 + \frac{z}{s} + \left(\frac{z}{s}\right)^2 + \cdots + \frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}}\right).$$

Therefore,

$$\frac{1}{s-z} = \frac{1}{s} + \frac{z}{s^2} + \frac{z^2}{s^3} + \cdots + \frac{z^{N-1}}{s^N} + \frac{z^N}{(s-z)s^N}.$$

Hence

$$f(z) = \frac{1}{2\pi i}\int_{C_{R_1}} f(s)\left\{\frac{1}{s} + \frac{z}{s^2} + \cdots + \frac{z^{N-1}}{s^N} + \frac{z^N}{(s-z)s^N}\right\} ds.$$

Now, the general Cauchy Integral formula says that

$$f^{(n)}(0) = \frac{n!}{2\pi i}\int_{C_{R_1}} \frac{f(s)}{s^{n+1}}\, ds.$$

Therefore,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{C_{R_1}} \frac{f(s)}{s}\, ds + \frac{z}{2\pi i}\int_{C_{R_1}} \frac{f(s)}{s^2}\, ds + \cdots \\ &\quad + \frac{z^{N-1}}{2\pi i}\int_{C_{R_1}} \frac{f(s)}{s^N} ds + \frac{1}{2\pi i}\int_{C_{R_1}} \frac{z^N}{(s-z)s^N}\, ds \\ &= f(0) + \frac{f'(0)}{1!}z + \frac{f^{(2)}(0)}{2!}z^2 + \cdots + \frac{f^{(N-1)}(0)}{(N-1)!}z^{N-1} + \rho_N(z), \end{aligned}$$

where

$$\rho_N(z) = \frac{1}{2\pi i}\int_{C_{R_1}} \frac{z^N f(s)}{(s-z)s^N}\, ds.$$

To complete our proof, it suffices to show that $\rho_N(z) \to 0$ as $N \to \infty$. Recall that $r < R_1$. Hence,

$$|s-z| \geq |s| - |z| = R_1 - r.$$

This implies that

$$|\rho_N(z)| \leq \frac{r^N}{2\pi}\frac{M_1}{(R_1-r)R_1^N}2\pi R_1 = \frac{M_1 R_1}{R_1 - r}\left(\frac{r}{R_1}\right)^N,$$

where

$$M_1 = \max_{s\in C_{R_1}} |f(s)|.$$

But $\frac{r}{R_1} < 1$ and therefore,

$$\lim_{N\to\infty}\left(\frac{r}{R_1}\right)^N = 0.$$

Hence, $\rho_N(z) \to 0$ when $N \to 0$. Thus, for each point in $B(0; R)$,

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f^{(2)}(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \cdots .$$

This special case of series is known as the Maclaurin series of $f(z)$. Setting $f^{(0)}(z) = f(z)$, we may rewrite the above series as

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k.$$

We now prove the Taylor series expansion of $f(z)$. Suppose $f(z)$ is analytic in $|z - z_0| < R$. Then $g(z) := f(z + z_0)$ is analytic in $B(0; R)$. Therefore

$$g(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} z^k.$$

But

$$g^{(k)}(0) = f^{(k)}(0 + z_0) = f^{(k)}(z_0).$$

Therefore,

$$f(z + z_0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} z^k, |z| < R.$$

Replacing z by $z - z_0$, we find that

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k,$$

which is (6.1), with $|z - z_0| < R$. □

DEFINITION 6.2 The largest R for which a power series

$$S(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

is convergent for all $|z - z_0| < R$ is called the radius of convergence of $S(z)$.

There are several ways of computing the radius of convergence of a given power series. We quote two of them.

THEOREM 6.2 Suppose $L = \lim_{k\to\infty} |C_k|^{1/k}$ exists.

(a) If $L = 0$, then

$$\sum_{n=0}^{\infty} C_n z^n$$

converges for all z.

(b) If $L = \infty$, then the series

$$\sum_{n=0}^{\infty} C_n z^n$$

converges for $z = 0$ only.

(c) If $0 < L < \infty$ then

$$\sum_{n=0}^{\infty} C_n z^n$$

converges for $|z| < \dfrac{1}{L}$ and diverges for $|z| > \dfrac{1}{L}$.

The above result is true if we replace $\lim_{k\to\infty} \ |C_k|^{1/k}$ by $\lim_{k\to\infty} \ |C_{k+1}/C_k|$.

EXAMPLE 6.1 The radius of convergence of $\sum_{k=1}^{\infty} z^k$ is 1. This is because the series converges to $\dfrac{1}{1-z}$ for $|z| < 1$.

EXAMPLE 6.2 The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is convergent everywhere. To show this, we compute

$$\lim_{k\to\infty} |C_{k+1}/C_k| = \lim_{k\to\infty} \frac{1}{k+1} = 0.$$

By the above result, we conclude that the series is convergent for all z. This function turns out to be e^z.

EXAMPLE 6.3 Show that

$$\frac{1}{z^2} = \sum_{k=0}^{\infty} (k+1)(z+1)^k, \quad |z+1| < 1.$$

Solution

We first write $1/z^2$ as $1/(1-(z+1))^2$ and then use the power series expansion for $1/(1-u)^2$.

We have seen in this section that if f is analytic at z_0 then f can be expanded as a convergent power series about z_0 in $B(z_0;r)$ for some $r>0$, namely,

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

If $a_j=0$ for $j=0,1,\cdots,m$, and $a_{m+1}\neq 0$, then we say that f has a zero of order m at z_0. When $m=1$, then we say that f has a simple zero at $z=z_0$. It can be shown directly that z_0 is a simple zero of an analytic function f if and only if $f'(z_0)\neq 0$.

6.3 Laurent Series

If a function f is not analytic at a point z_0, we cannot apply Taylor's Theorem at that point. It is, however, possible to find a series representation for $f(z)$ involving both positive and negative powers of $z-z_0$. A series representation of $f(z)$ that involves negative powers of $z-z_0$ is called a Laurent series of $f(z)$ about z_0.

EXAMPLE 6.4 Find the Laurent expansion of

$$f(z)=\frac{1+2z}{z^2-z^3}$$

about $z=0$.

Solution

The expansion is

$$\begin{aligned}
f(z) &= \frac{1}{z^2}\left(\frac{1+2z}{1-z}\right)\\
&= \frac{1}{z^2}(1+2z)(1+z+z^2+\cdots), \quad 0<|z|<1,\\
&= \frac{1}{z^2}\left(1+2z+z+2z^2+z^2+2z^3+\cdots\right)\\
&= \frac{1}{z^2}+\frac{3}{z}+3+3z+\cdots.
\end{aligned}$$

This series expansion is convergent on $0<|z|<1$.

We will only discuss the case when $z_0=0$.

THEOREM 6.3 If f is analytic in the annulus

$$A = \{z \in \mathbf{C} | R_1 < |z| < R_2\},$$

then f has a Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k,$$

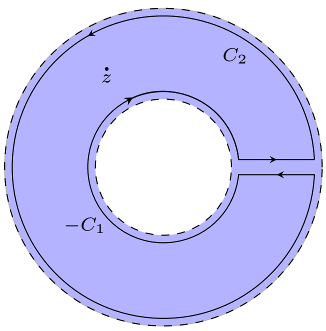
where

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{k+1}} d\zeta$$

and $C = C(0; R)$ with $R_1 < R < R_2$.

Sketch of proof
The proof of this theorem is similar to that of Theorem 6.1. Suppose $R_1 < r_1 < r_2 < R_2$. We consider the expansion of $1/(z - s)$, with $s \in \{z||z| = r_1\}$ or $s \in \{z||z| = r_2\}$ in two ways.

Let C be the contour $C(0; r_2)$ traversed in counterclockwise direction, and then meet $C(0; r_1)$, traversed in the clockwise direction and finally return to meet $C(0; r_2)$ (see the following diagram).



The result is a simple closed curve that encloses z. Suppose C_1 and C_2 are the respective paths along $C(0; r_1)$ and $C(0; r_2)$ traversed in the anticlockwise direction. Then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta = I_1 + I_2. \end{aligned}$$

For the first integral, expand $1/(1-z/\zeta)$ since or $|z/\zeta|<1$. The result is the same as that for Taylor's series, i.e.

$$I_1 = \sum_{n=0}^{\infty} a_n z^n$$

with

$$a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta.$$

For the second integral, we expand $1/(1-\zeta/z)$ instead of $1/(1-z/\zeta)$ since $|\zeta/z|<1$. We have

$$\begin{aligned} I_2 &= \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{z(1-\zeta/z)} d\zeta \\ &= \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{z}\left\{1+\frac{\zeta}{z}+\cdots+\left(\frac{\zeta}{z}\right)^{N-1}+\left(\frac{\zeta}{z}\right)^{N}\frac{1}{(1-\zeta/z)}\right\} d\zeta \\ &= \sum_{j=1}^{N-1}\frac{b_j}{z^j}+\sigma_N, \end{aligned}$$

where

$$b_j = \frac{1}{2\pi i}\int_{C_1} f(\zeta)\zeta^{j-1} d\zeta$$

and

$$\sigma_N = \frac{1}{2\pi i}\int_{C_1}\left(\frac{\zeta}{z}\right)^N \frac{f(\zeta)}{z-\zeta} d\zeta.$$

Now,

$$|\sigma_N| \le \frac{1}{2\pi} 2\pi r_1 \left|\frac{r_1}{z}\right|^N \frac{M}{|z|-r_1},$$

where $|f(z)|\le M$ on C_1. Hence, $\sigma_N \to 0$ as $N\to 0$ since $|r_1/z|<1$.

Observe that C_1 and C_2 can now be replaced by a common circle $C=C(0;R)$ with $R_1<R<R_2$. Next, if we set $b_j=a_{-j}$ then the formula

$$a_{-j} = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta^{-j+1}} d\zeta$$

holds. This means that for all $j\in \mathbf{Z}$,

$$a_j = \frac{1}{2\pi i}\int_{C} \frac{f(\zeta)}{\zeta^{j+1}} d\zeta.$$

This completes the proof of the theorem. □

COROLLARY 6.4 If f is analytic in the annulus $R_1 < |z - z_0| < R_2$, then f has a power series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

where

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}} dz$$

and $C = C(z_0; R)$ with $R_1 < R < R_2$.

We now give more examples of Laurent series expansions.

EXAMPLE 6.5 The function $f(z) = \dfrac{1}{(z-1)^2}$ is analytic in $0 < |z - 1| < \infty$. The Laurent expansion of $f(z)$ about $z = 1$ is just $\dfrac{1}{(z-1)^2}$.

EXAMPLE 6.6 The Laurent series expansion of $\dfrac{e^z}{z^2}$ about $z = 0$ is

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots .$$

The region for which this is valid is $0 < |z| < \infty$.

EXAMPLE 6.7 $e^{1/z}$ has Laurent series expansion about $z = 0$ as

$$e^{1/z} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{z^n}, 0 < |z| < \infty.$$

Note that the b_n in this case are non-zero for infinitely many n.

EXAMPLE 6.8 Find the first 3 non-zero terms of the Laurent series expansion of

$$\frac{\sin z}{z^3(1 - z)}$$

about $z = 0$.

DEFINITION 6.3 We say that z_0 is an isolated singularity of $f(z)$ if $f(z)$ is analytic on $B(z_0; r) - \{z_0\}$ for some $r > 0$.

We now classify isolated singularities according to the Laurent series expansion of $f(z)$ about z_0. Suppose the Laurent series expansion of $f(z)$ about z_0 is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

If $a_n = 0$ for $n < 0$, then we say that z_0 is a removable singularity of $f(z)$.

If $a_n = 0$ for $n < -m$ for some negative integer $-m$, then we say that $f(z)$ has a pole of order m at z_0. The function $1/(z - z_0)^m$ is a function with pole of order m at z_0.

If $a_n \neq 0$ for infinitely many negative integers n, then we say that z_0 is an essential singularity of $f(z)$. The function $e^{1/z}$ is a function with essential singularity at $z = 0$.

EXAMPLE 6.9 Find all functions $f(z)$ which is analytic at all z except at 0 satisfying the condition that for all non-zero $z \in \mathbf{C}$,

$$|f(z)| \leq \frac{1}{|z|^{1/2}} + |z|^{1/2}.$$

Solution

Let $A = \{z | 0 < |z| < R_2\}$. We know that if $z \in A$,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k,$$

where

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{k+1}} \, d\zeta$$

whenever C is a simple closed curve enclosing the origin. Suppose $k \geq 1$. Let $C = C(0; r)$. Then by ML-formula and the bound for $|f(z)|$,

$$|a_k| = \left| \frac{1}{2\pi i} \int_{C(0;r)} \frac{f(\zeta)}{\zeta^{k+1}} \, d\zeta \right| \leq \frac{1}{2\pi} 2\pi r \left(\frac{\frac{1}{\sqrt{r}} + \sqrt{r}}{r^{k+1}} \right).$$

The right hand side tends to 0 as $r \to \infty$. Therefore,

$$a_k = 0$$

for $k \geq 1$.

Suppose $k \leq -1$. Then

$$|a_k| = \left| \frac{1}{2\pi i} \int_{C(0;\epsilon)} \frac{f(\zeta)}{\zeta^{k+1}} \, d\zeta \right| \leq \frac{1}{2\pi} 2\pi\epsilon \left(\frac{\frac{1}{\sqrt{\epsilon}} + \sqrt{\epsilon}}{\epsilon^{k+1}} \right).$$

The right hand side tends to 0 as $\epsilon \to 0$. Therefore,

$$a_k = 0$$

for $k \leq -1$. Therefore, $f(z) = a_0$ and it must be a constant.

6.4 Absolute Convergence, Uniform Convergence and continuity of power series

We say that a series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is convergent if $\lim_{N\to\infty} \sum_{n=0}^{N} a_n (z - z_0)^n$ has a limit. If

$$S^*(z) = \sum_{n=0}^{\infty} |a_n (z - z_0)^n|$$

converges then we say that the series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is absolutely convergent.

THEOREM 6.5 If

$$S(z) = \sum_{k=0}^{\infty} a_n (z - z_0)^n$$

converges when $z = z_1$, $(z_1 \neq 0)$, it is absolutely convergent for every value of z such that $|z - z_0| < |z_1 - z_0|$.

Proof

Let

$$r = \left| \frac{z - z_0}{z_1 - z_0} \right| < 1.$$

Since

$$\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$$

converges,

$$\lim_{n\to\infty} |a_n(z_1 - z_0)^n| = 0.$$

This implies that for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbf{Z}^+$ such that

$$|a_n(z_1 - z_0)^n| \leq \epsilon$$

whenever $n \geq N_\epsilon$. Set $\epsilon = 1$. Then for $n \geq N_1$,

$$|a_n(z_1 - z_0)^n| \leq 1.$$

For $n \leq N_1$,

$$|a_n(z_1 - z_0)^n| \leq \max_{0\leq j\leq N_1} |a_j(z_1 - z_0)^j| = M.$$

Therefore,

$$|a_n(z_1 - z_0)^n| \leq \max(1, M)$$

for all integers $n \geq 0$.

Let

$$S_\ell^* := \sum_{k=0}^{\ell} |a_k(z - z_0)^k|.$$

Then

$$\begin{aligned} |S_m^* - S_n^*| &= \sum_{k=n+1}^{m} |a_k(z - z_0)^k| \\ &= \sum_{k=n+1}^{m} |a_k(z_1 - z_0)^k| \left|\frac{z - z_0}{z_1 - z_0}\right|^k \\ &\leq \max(1, M) \sum_{k=n+1}^{m} r^k. \end{aligned}$$

But the sequence of partial sums $\{G_\ell\}$ where $G_\ell := \sum_{k=0}^{\ell} r^k$ is a Cauchy sequence. Hence for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbf{Z}^+$ such that

$$|G_m - G_n| < \frac{\epsilon}{\max(1, M)} \quad \text{whenever } m > n \geq N_\epsilon.$$

Therefore,

$$\sum_{k=n+1}^{m} |a_k(z - z_0)^k| < \epsilon$$

whenever $m > n \geq N_\epsilon$ which implies that $S(z)$ is absolutely convergent. □

In general the rate at which $S_N(z) = \sum_{n=0}^{N} a_n(z-z_0)^n$ converges to $S(z)$ depends on z, i.e., for any $\epsilon > 0$,

$$|S_N(z) - S(z)| < \epsilon, \quad \text{whenever} \quad N \geq N_\epsilon(z).$$

However, if we choose z such that $|z - z_0| \leq R_1 < R$, where R is the radius of convergence of $S(z)$, then there exist an N_ϵ which will work for all z in $|z-z_0| \leq R_1 < R$. When N_ϵ is independent of z, we say that $S(z)$ is uniformly convergent. We also say that $S_N(z)$ converges uniformly to $S(z)$ if $|z| \leq R_1 < R$.

THEOREM 6.6 The series

$$S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is uniformly convergent for $|z - z_0| \leq R_1 < R$, where R is the radius of convergent.

Proof
Let u be fixed and $|u - z_0| = R_1$. Let $z \in \overline{B(z_0; R_1)}$. Let

$$\rho_N(z) = \lim_{m\to\infty} \sum_{n=N}^{m} a_n(z-z_0)^n = S(z) - \sum_{n=0}^{N-1} a_n(z-z_0)^n.$$

Now,

$$\left|\sum_{n=N}^{m} a_n(z-z_0)^n\right| \leq \sum_{n=N}^{m} |a_n||z-z_0|^n \leq \sum_{n=N}^{m} |a_n||u-z_0|^n. \tag{6.1}$$

Now let r be such that $R_1 < r < R$ and $|w - z_0| = r$. Note that

$$\sum_{n=0}^{\infty} a_n(w-z_0)^n$$

converges since $|w - z_0| = r < R$. Hence, by Theorem 6.5,

$$\sum_{n=0}^{\infty} a_n(u-z_0)^n$$

converges absolutely. This implies that for every $\epsilon > 0$, there exists a positive integer N_ϵ such that

$$\sum_{n=N}^{m} |a_n(u-z_0)^n| < \epsilon \quad \text{whenever} \quad m \geq N_\epsilon.$$

Therefore, by (6.1), we deduce that

$$\left|\sum_{n=N}^{m} a_n(z-z_0)^n\right| < \epsilon \quad \text{whenever} \quad m \geq N_\epsilon,$$

with N_ϵ independent of z. Hence, $S(z)$ converges uniformly in $\overline{B(z_0; R_1)}$. □

THEOREM 6.7 A power series

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

represents a continuous function $S(z)$ on $B(z_0; R)$, where R is the radius of convergence of the power series.

Proof

Let $w \in B(z_0; R)$ and choose $\delta > 0$ such that $B(w; \delta) \subset \overline{B(z_0; R_1)}$ for some $R_1 < R$. Then the series $S(z)$ converges uniformly in $\overline{B(w; \delta)}$ and this implies that for any $\epsilon > 0$, there exists a positive integer N_ϵ such that

$$\left| \sum_{N+1}^{m} a_n(z-z_0)^n \right| < \frac{\epsilon}{3} \text{ whenever } m > N \geq N_\epsilon \tag{6.2}$$

and the above holds for any $z \in \overline{B(w; \delta)}$. Write $S(z) = S_{N_\epsilon}(z) + \rho_{N_\epsilon}(z)$, with

$$\rho_N(z) = \lim_{m \to \infty} \sum_{n=N+1}^{m} a_n(z-z_0)^n.$$

Now, write

$$\begin{aligned} |S(z) - S(w)| &= |S_{N_\epsilon}(z) - S_{N_\epsilon}(w) + \rho_{N_\epsilon}(z) - \rho_{N_\epsilon}(w)| \\ &\leq |S_{N_\epsilon}(z) - S_{N_\epsilon}(w)| + |\rho_{N_\epsilon}(z)| + |\rho_{N_\epsilon}(w)|. \end{aligned} \tag{6.3}$$

By (6.2),

$$|\rho_{N_\epsilon}(z)| < \frac{\epsilon}{3} \quad \text{and} \quad |\rho_{N_\epsilon}(w)| < \frac{\epsilon}{3}. \tag{6.4}$$

Next, since $S_{N_\epsilon}(z)$ is a polynomial in z, $S_{N_\epsilon}(z)$ is continuous at $z = w$. Therefore, given $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that

$$|S_{N_\epsilon}(z) - S_{N_\epsilon}(w)| < \frac{\epsilon}{3} \quad \text{whenever} \quad |z - w| < \delta_\epsilon. \tag{6.5}$$

By (6.4), (6.3) and (6.5), we conclude that

$$|S(z) - S(w)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

whenever $|z - w| < \min(\delta, \delta_\epsilon)$. This implies that $S(z)$ is continuous at w. □

6.5 Power series and Analytic functions

We have shown that if $f(z)$ is analytic at z_0, then $f(z)$ can be expressed as a series about z_0 in $\overline{B(z_0; r)}$ for some $r > 0$. In this section, we will show that a power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

which is convergent in $B(z_0; R)$ is an analytic function on $B(z_0; R)$.

THEOREM 6.8 Let

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

be a convergent power series about z_0 on $B(z_0; R)$. Let C be a simple closed curve contained in $B(z_0; R)$ and $g(z)$ be a continuous function on C. Then

$$\int_C g(z) \sum_{k=0}^{\infty} a_k (z - z_0)^k \, dz = \sum_{k=0}^{\infty} a_k \int_C g(z)(z - z_0)^k \, dz.$$

Proof
Since $g(z)$ is continuous on C, there exists a positive real number M^* such that $|g(z)| < M^*$. Let $\epsilon > 0$. There exists a positive integer N_ϵ such that

$$\left| \sum_{k=n+1}^{\infty} a_k (z - z_0)^k \right| < \frac{\epsilon}{LM^*}$$

for all $n \geq N_\epsilon$. Then

$$\begin{aligned} \left| \int_C g(\zeta) S(\zeta) d\zeta - \sum_{k=0}^{n} a_k \int_C g(\zeta)(\zeta - z_0)^k d\zeta \right| &= \left| \int_C g(\zeta) \sum_{k=n+1}^{\infty} a_k (\zeta - z_0)^k d\zeta \right| \\ &< LM^* \frac{\epsilon}{LM^*} = \epsilon \end{aligned}$$

whenever $n \geq N_\epsilon$. Hence the result.

□

First, we observe that $S(z)$ is continuous on $B(z_0; R)$ by Theorem 6.7. When $g(z) = 1$, we know that

$$\int_C (z - z_0)^n dz = 0$$

for any simple closed curve and therefore, if $S(z)$ is a convergent power series, then by Theorem 6.8,

$$\int_C S(z) dz = 0$$

for any closed curve in $|z - z_0| \leq R_1 < R$. In particular,

$$\int_{\partial T} S(z) dz = 0$$

for any triangular contour ∂T contained in $B(z_0;R)$. By Morera's Theorem, we conclude the following:

THEOREM 6.9 Let $S(z)$ be a convergent power series on $B(z_0;R)$. Then $S(z)$ is an analytic function on $B(z_0;R)$.

EXAMPLE 6.10 Let

$$f(z) = \begin{cases} \dfrac{\sin z}{z} & \text{when} \quad z \neq 0 \\ 1 \quad \text{when} \quad z = 0. \end{cases}$$

The power series for $\sin z/z$ is

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots =: S(z)$$

and this series is convergent for all z except at $z = 0$. But $S(0) = f(0)$. Therefore, $f(z)$ is the power series $S(z)$ and it is entire since the power series $S(z)$ converges for all $z \in \mathbf{C}$.

Theorem 6.8 shows that if $g(\zeta) = 1$ and C is the line segment from 0 to z and

$$F(z) = \int_{[0,z]} f(\zeta)\, d\zeta,$$

then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

implies that

$$F(z) = \sum_{n=0}^{\infty} a_n \int_{[0,z]} \zeta^n d\zeta = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}.$$

In other words, we can obtain the power series representation of the primitive of $f(z)$ from the power series representation of $f(z)$. We next determine the power series of $f'(z)$ from the power series representation of $f(z)$.

THEOREM 6.10 A convergent power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

can be differentiated term by term. That is, at each point z interior to the circle of convergence of that series,

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Proof
Let $B(z_0; R)$ be the region for which $S(z)$ is convergent. Let $z \in B(z_0; R)$ and let $C \subset B(z_0; R)$ be a simple closed curve enclosing z and suppose C is traversed in the counterclockwise direction. We let

$$g(\zeta) = \frac{1}{2\pi i} \frac{1}{(\zeta - z)^2}$$

at each point of $\zeta \in C$. Since $g(\zeta)$ is continuous on C, Theorem 6.8 implies that

$$\int_C g(\zeta) S(\zeta) d\zeta = \sum_{n=0}^{\infty} a_n \int_C g(\zeta)(\zeta - z_0)^n d\zeta.$$

Now, $S(\zeta)$ is analytic inside and on C and this enables us to write

$$\int_C g(\zeta) S(\zeta) d\zeta = \frac{1}{2\pi i} \int_C \frac{S(\zeta)}{(\zeta - z)^2} d\zeta = S'(z).$$

Furthermore,

$$\int_C g(\zeta)(\zeta - z_0)^n d\zeta = \frac{1}{2\pi i} \int_C \frac{(\zeta - z_0)^n}{(\zeta - z)^2} d\zeta = n(z - z_0)^{n-1}, n = 1, 2, \cdots.$$

Thus, we find that

$$S'(z) = \sum_{n=1}^{\infty} a_n n (z - z_0)^{n-1}.$$

□

As an application of the above theorem, we deduce that the power series expansion

$$\sin z = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^{2k-1}}{(2k-1)!}$$

leads to

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}.$$

One can also use the theorem to deduce from

$$\frac{1}{1 - z} = \sum_{k=0}^{\infty} z^k$$

that for positive integer n,

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!}(-z)^k.$$

We have seen in Theorem 6.1 that if $f(z)$ is analytic at z_0, then $f(z)$ can be written as a Taylor series on $B(z_0;r)$ for some $r > 0$. We now use Theorem 6.8 to show that the power series expansion of an analytic function $f(z)$ about z_0 is unique on $B(z_0;r)$. Suppose

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k$$

is another power series expansion of $f(z)$ at z_0. Let $C = C(z_0;\delta)$ be contained in $B(z_0;r)$. Let

$$g(\zeta) = \frac{1}{2\pi i}\frac{1}{(\zeta - z_0)^{N+1}}$$

in Theorem 6.8. Then

$$\begin{aligned}\frac{1}{2\pi i}\int_C f(\zeta)\frac{1}{(\zeta-z_0)^{N+1}}d\zeta &= \frac{1}{2\pi i}\int_C S(\zeta)\frac{1}{(\zeta-z_0)^{N+1}}d\zeta \\ &= \sum_{k=0}^{\infty} a_k \left(\frac{1}{2\pi i}\int_C \frac{(\zeta-z_0)^k}{(\zeta-z_0)^{N+1}}d\zeta\right).\end{aligned}$$

This implies that

$$a_N = \frac{f^{(N)}(z_0)}{N!}.$$

Hence, the power series expansion of $f(z)$ about z_0 must coincide with the Taylor series expansion of $f(z)$ about z_0 for every $z \in B(z_0;r)$.

We have shown that if R is the radius of convergent of a power series about z_0, then the power series converges uniformly in $\overline{B(z_0;R_1)}$ with $R_1 < R$. In a similar way, we can prove that if

$$T(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

is convergent on $|z-z_0| > r$, then $T(z)$ is converges uniformly in $|z-z_0| \geq r$, with $r_1 > r$. Hence, we would also have

$$\int_C g(\zeta)\sum_{n=1}^{\infty}\frac{b_n}{(\zeta-z_0)^n}\,d\zeta = \sum_{n=1}^{\infty} b_n \int_C \frac{g(\zeta)}{(\zeta-z_0)^n}\,d\zeta,$$

for any $g(z)$ continuous on C, and C contained in $|z-z_0| > r$, the region of convergence of

$$\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}.$$

Combining the series involving positive and negative powers of $(z - z_0)$ we conclude that if

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is convergent in $r < |z - z_0| < R$ and C is in the region, then

$$\int_C g(\zeta) f(\zeta) d\zeta = \sum_{n=-\infty}^{\infty} a_n \int_C g(\zeta)(\zeta - z_0)^n d\zeta.$$

This would imply the Laurent series of a function $f(z)$ in $r < |z - z_0| < R$ is unique.

7 Uniqueness Theorem and Maximum Modulus Principle

7.1 Uniqueness Theorem for Power series

Let $f(z)$ be an analytic function on $B(z_0; r)$. If there exists positive $r < R$ such that $f(z) = 0$ on $B(z_0; \epsilon)$, then $f^{(n)}(z) = 0$ on $B(z_0; \epsilon)$. Then the formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C(z_0;r)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta = 0$$

for all non-negative integers n. Since

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

on $B(z_0; r)$ and $f^{(n)}(z_0) = 0$, we deduce the following:

THEOREM 7.1 If $f(z)$ is an analytic function on $B(z_0; r)$ and $f(z) = 0$ on $B(z_0; \epsilon)$ for some $0 < \epsilon < r$ then

$$f(z) = 0$$

on $B(z_0; r)$.

From Theorem 7.1, we know that if $f(z)$ is analytic on $B(u; r)$ and $f(z)$ vanishes on $B(u; \epsilon)$ for some $0 < \epsilon < r$, then $f(z)$ vanishes on $B(u; r)$.

In our next theorem, we show a stronger version of Theorem 7.1. We show that $f(z)$ vanishes on $B(u; r) \subset D$ when there is a sequence $\{w_k\} \subset B(u; r)$ which converges to u for which $f(w_k) = 0$ for $k \in \mathbf{Z}^+$.

THEOREM 7.2 Suppose $f(z)$ is analytic on a region D and is zero at all points of a sequence $\{w_k\}_{k=1}^{\infty}$ which converges to $u \in D$. Then there exists a positive real number r such that $B(u; r) \subset D$ and $f(z) = 0$ for all $z \in B(u; r)$.

Proof

Since $f(z)$ is analytic at u, $f(z)$ has a power series expansion representation $S(z)$

given by

$$S(z) = \sum_{n=0}^{\infty} a_n (z-u)^n$$

for $z \in B(u;\epsilon)$. Since a convergent power series $S(z)$ is continuous,

$$a_0 = S(z_0) = \lim_{k\to\infty} S(w_k) = 0.$$

Next,

$$a_1 = \frac{S(z)}{z-u} - a_2(z-u) - a_3(z-u)^2 - \cdots .$$

Let $z = w_k$. We find that

$$a_1 = -a_2(w_k - u) - a_3(w_k - u)^2 - \cdots$$

since $S(w_k) = 0$. Hence,

$$a_1 = \lim_{k\to\infty} \left(-a_2(w_k - u) - a_3(w_k - u)^2 - \cdots\right) = 0$$

since the power series is continuous. Suppose $a_i = 0$, for $0 \le i \le n$. Then

$$a_{n+1} = \lim_{k\to\infty} \left(-a_{n+2}(w_k - u) - a_{n+3}(w_k - u)^2 - \cdots\right) = 0.$$

Therefore, $a_i = 0$ for all $i \in \mathbf{N}$. This implies that $S(z) = 0$ for all $z \in B(u;\epsilon)$. Now, $f(z)$ vanishes on $B(u;\epsilon)$ implies that $f(z)$ vanishes on $B(u;r)$ whenever $B(u;r) \subset D$ by uniqueness of power series. □

Remark 7.1 In the above proof, we have used the fact that if $f(z)$ is a continuous function in D and $\{w_k\}_{k=1}^{\infty}$ is a sequence in D such that

$$\lim_{k\to\infty} w_k = u$$

then

$$\lim_{k\to\infty} f(w_k) = f(u).$$

In Theorem 7.2, we show that the vanishing of $f(z)$ on $B(u;r)$ follows from the existence of a sequence of zeroes of $f(z)$ approaching $u \in D$. Our next result shows that with the hypothesis of Theorem 7.2, we can conclude that $f(z) = 0$ on D (instead of its vanishing on $B(u;r) \subset D$ for some $r > 0$).

DEFINITION 7.1 Let f be continuous on a region D. We say that $u \in D$ is a limit of zeroes of f if there exists a sequence $\{w_k\}$ such that $w_k \to u$, with $f(w_k) = 0$.

THEOREM 7.3 Suppose $f(z)$ is analytic in a region D and $f(w_n) = 0$ where $w_n \to u \in D$. Then $f(z) = 0$ in D.

Proof
Step 1: Let

$$A = \{z \in D : z \text{ is a limit of zeros of } f\}.$$

Note that $A \neq \phi$ since $u \in A$. Let $B = D \backslash A$. Note that $D = A \cup B$ and $A \cap B = \phi$.
Step 2: We now show that A is open. Let $z' \in A$. Since z' is a limit of zeroes of $f(z)$, there is a sequence $\{w_k\}$ with $w_k \to z'$ such that $f(w_k) = 0$ for all non-negative integers k. By Theorem 7.2, we deduce that $f(z)$ vanishes on $B(z'; r) \subset D$. Now, each $w \in B(z'; r)$ is a limit of zeros of $f(z)$ since $f(w) = 0$ for all $w \in B(z'; r)$. Therefore $B(z'; r) \subset A$, which implies that A is an open set in $\mathbf{C}$.
Step 3: Next we show that B is open. Suppose $z^* \in B$. Then since z^* is not a limit of zeros of f, there exist an open set $B(z^*, \delta)$ for which $f(z) \neq 0$ for all $z \in B(z^*, \delta)$ except possibly that $f(z^*) = 0$. This implies that none of the elements in the open set is a limit of zeros of f and so they are all in B. Hence, B is open. But D is connected and therefore cannot be a union of two open sets. Since we have noted that A is non-empty by Step 1, this implies that $B = \phi$. Hence $A = D$ and every element $z \in D$ is a limit of zeros of f and hence $f(z) = 0$ in D. □

COROLLARY 7.4 If $f(z)$ and $g(z)$ are analytic in a region D and agree at a set of points $\{w_k\}$ with $\lim_{k\to\infty} w_k = u$ where $u \in D$, then $f(z) = g(z)$ in D.

This is known as the Uniqueness Theorem for analytic functions. This result explains why series expansions for real functions such as e^x, $\sin x$ etc are the same as those for e^z, $\sin z$ etc..

EXAMPLE 7.1 The formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for $x \in \mathbf{R}$ implies, by the Uniqueness Theorem, that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

for $z \in \mathbf{C}$. These two formulas look the same but they are not. For example, when $z = iy, y \in \mathbf{R}$,

$$e^{iy} = \cos y + i \sin y = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!},$$

giving power series expansions for $\cos y$ and $\sin y$. Note that these series cannot be obtain from the power series expansion of e^x.

EXAMPLE 7.2 Let f and g be analytic functions on $B(0;1)$ with $g(z) \neq 0$ for all $z \in B(0;1)$. Suppose for all integers $n \geq 1$,

$$\frac{f'(1/n)}{f(1/n)} = \frac{g'(1/n)}{g(1/n)}.$$

Show that f/g is a constant on $B(0;1)$.

Solution

Let $G = (f/g)'$. Then

$$G(1/n) = \frac{f'(1/n)g(1/n) - f(1/n)g'(1/n)}{g^2(1/n)} = 0.$$

Since $\lim_{n\to\infty} 1/n = 0 \in B(0;1)$ which implies that 0 is the limit of zeroes of $G(z)$, we conclude by Uniqueness Theorem, $G(z) = 0$ on $B(0;1)$. Since $G = (f/g)'$, we conclude that f/g is a constant on $B(0;1)$.

We end this section with a final application of the Uniqueness Theorem.

EXAMPLE 7.3 Show that if f is a non-constant analytic on a region D and $F \subset D$ is a closed and bounded set, then F contains finitely many zeroes of f.

Solution

Suppose the contrary. Then F contains infinitely many zeroes of f. Since F is closed and bounded, it is compact. Therefore, F can be covered by finitely many open balls of radius 1. Since there are infinitely many zeroes of f in F and the number of open balls is finite, there exists an open ball (may be more than one such open balls) $\mathcal{O}_1$ which contains infinitely many zeroes of f. The set $F_1 = \overline{\mathcal{O}_1} \cap F$ is a closed subset of F. Next, F_1 is compact. Therefore there are finitely many open balls of radius 1/2 that cover F_1. We choose one such open balls $\mathcal{O}_2$ which contains infinitely many zeroes of f. Let $F_2 = \overline{\mathcal{O}_2} \cap F$. By continuing with this process and reducing the radius of the ball by a factor of $1/2$ at each stage, we obtain a nested closed sets

$$\cdots \subset F_2 \subset F_1 \subset F.$$

Note that since $\mathcal{O}_j$ is an open ball of radius $1/2^j$, the diameter of F_j tends to 0 as j tends to infinity. By Cantor's intersection Theorem,

$$\bigcap_j F_j = \{w\}.$$

Now, in each F_j, we choose a distinct zero of f and called in z_j. Note that

$$|z_j - w| \leq 1/2^j$$

since $w \in F_j$. Therefore z_j tends to w as j tends to infinity. Finally,

$$0 = \lim_{j\to\infty} f(z_j) = f(w).$$

Therefore w is a limit of zeroes of f and this implies that f is identically 0, which contradicts to the assumption that f is non-constant. Hence, there are only finitely many zeroes of f in F.

EXAMPLE 7.4 (Bak-Newman (First edition), p. 74, Problem 6) Show that if f is entire and $|f(z)| \geq |z|^N$ for sufficiently large z. then f must be a polynomial of degree at least N.

Solution

Observe that if w is a zero of $f(z)$, then from the power series expansion of f at w, we find that

$$f(z) = (z-w)^m \sum_{k=m+1}^{\infty} a_k (z-w)^k.$$

So we may write

$$f(z) = (z-w)^m f_1(z),$$

where $f_1(w) \neq 0$.

Let $R > 0$ be sufficiently large so that for $|z| \geq R$, $|f(z)| \geq |z|^N$. By Uniqueness Theorem, we know that the number of zeroes of f is finite in $B(0;R)$ (see the previous example for more details). Let $z_1, z_2, \cdots, z_\ell$ (not necessarily distinct) be all the zeroes of f in $B(0;R)$. By the remark in the beginning of the proof, we may write

$$f(z) = (z-z_1)\cdots(z-z_\ell)g(z)$$

where $g(z) \neq 0$ in $B(0;R)$. Let

$$P(z) = (z-z_1)\cdots(z-z_\ell).$$

We have observed that the function $g(z) \neq 0$ in $B(0;R)$. If $g(v)=0$ for some v such that $|v| \geq R$, then $f(v)=P(v)g(v)=0$. This means that $|f(v)|=0$, which is a contradiction since $|f(v)| \geq |v|^N > 0$ since $|v| \geq R$. Therefore, $g(z)$ is a non-vanishing entire function and $h(z)=1/g(z)$ is entire.

Now, for $|z| \geq R$, $|P(z)| \leq A|z|^{\ell}$ for some $A>0$ and $|f(z)| \geq |z|^N$. Therefore,

$$|h(z)| \leq A|z|^{\ell-N},$$

for $|z| \geq R$. For $|z| \leq R$,

$$|h(z)| \leq M$$

since h is continuous. Hence, $h(z)$ is an entire function which satisfies an inequality of the form

$$|h(z)| \leq C|z|^{\ell-N}+D, C>0, D>0,$$

and for all $z \in \mathbf{C}$. By extended Liouville Theorem, $h(z)$ is a polynomial of degree at most $\ell - N$. Now, $f(z)h(z)=P(z)$. Since $f(z)$ is entire, the polynomial $h(z)$ must divide $P(z)$ and therefore, $f(z)$ is a polynomial. The sum of the degrees of $f(z)$ and $h(z)$ is equal to the degree of $P(z)$. In other words,

$$m = \deg f(z) + \deg h(z) \leq \deg f(z) + m - N.$$

Therefore,

$$\deg f(z) \geq N.$$

7.2 Minimum Modulus Principle and Open Mapping Theorem

We begin this section by proving the Minimum Modulus Principle.

THEOREM 7.5 Let $f(z)$ be a non-constant analytic function in a region R and $f(z) \neq 0$ for all $z \in R$, then there is no point $\alpha \in R$ satisfying the relation $|f(\alpha)| \leq |f(z)|$ for all $z \in R$.

Proof
Since $f(z) \neq 0$ for all $x \in R$, the function $g(z)=1/f(z)$ is analytic in R. By maximum modulus principle, there does not exist $\alpha \in R$ such that $|g(z)| \leq |g(\alpha)|$ for all $z \in R$. This translates to the statement that there is no $\alpha \in R$ that satisfies the inequality $|f(\alpha)| \leq |f(z)|$ for all $z \in R$. □

Remark 7.2 The above theorem implies that if $f(z)$ is a non-constant analytic function in a bounded region R and continuous on ∂R, then the minimum of $|f(z)|$ must occur at ∂R.

We have seen that if $f(z)$ is an analytic function that is purely imaginary then $f(z)$ is a constant. This is a special case of an important result known as open mapping theorem.

THEOREM 7.6 The image of an open set under a nonconstant analytic mapping is an open set

Proof
We will give a proof due to C. Carathéodory. Our aim is to show that if X is an open set, then

$$f(X) = \{f(z) | z \in X\}$$

is an open set. Let $\beta \in f(X)$. Then there exists $\alpha \in X$ such that

$$f(\alpha) = \beta.$$

Since $\alpha \in X$ and X is open, there exists $R > 0$ such that

$$B(\alpha; R) \subset X.$$

By uniqueness theorem, there exists $0 < r < R$ such that for all $z \in C(\alpha; r)$, $f(z) \neq \beta$. This implies that there exists $\epsilon > 0$ such that

$$2\epsilon = \min_{z \in C(\alpha;r)} |f(z) - \beta|.$$

We will show that $B(\beta; \epsilon)$ is contained in $f(X)$ and this will imply that $f(X)$ is open. Let $u \in B(\beta; \epsilon)$. If $u \notin f(X)$ then $f(z) - u \neq 0$ for all $z \in \overline{B(\alpha; r)}$. This implies, by Remark 7.2, that the minimum of $|f(z) - u|$ must occur at some $v \in C(\alpha; r)$. In other words, for all $z \in B(\alpha; r)$,

$$|f(z) - u| > |f(v) - u|$$

for some $v \in C(\alpha; r)$. Now, for $z \in B(\alpha; r)$,

$$|f(z) - u| > |f(v) - u| = |f(v) - \beta + \beta - u| \geq |f(v) - \beta| - |\beta - u| \geq 2\epsilon - \epsilon = \epsilon, \quad (7.1)$$

where we have used the fact that the $\min_{z \in C(\alpha;r)} |f(v) - \beta| = |f(v) - \beta| = 2\epsilon$ and that $u \in B(\beta; \epsilon)$. Substituting $z = \alpha$ to the left hand side of (7.1) and observing that $|f(\alpha) - u| = |\beta - u| < \epsilon$, we conclude from (7.1) that

$$\epsilon > |\beta - u| = |f(\alpha) - u| > \epsilon,$$

which is a contradiction. Hence, $f(z) = u$ for some $z \in B(\alpha; r)$. In other words,

$$B(\beta; \epsilon) \subset f(B(\alpha; r)) \subset f(X)$$

and $f(X)$ is open.

□

7.3 Appendix: Polygonally connected and connected

In this appendix, we show that in **C**, polygonally connected is equivalent to connected. Recall that a set D is disconnected if there exists nonempty disjoint open sets A and B such that

$$D = A \cup B.$$

Otherwise, D is said to be connected.

THEOREM 7.7 An open set D is connected if and only if it is polygonally connected

Proof
Suppose D is connected. Let $u \in D$ and let

$$A = \{s \in D | s \text{ is connected to } u \text{ by a polygonal line in } D\}.$$

Let $B = D \backslash A$, i.e., every point in B is not connected to u by a polygonal line in D. Note that $A \neq \phi$ because $u \in A$. Also, $D = A \cup B$ and $A \cap B = \phi$.

We now show that A is open. If $z \in A$, then $B(z;r) \subset D$ for some $r > 0$ since D is open. But in $B(z;r)$, any two points are polygonally connected. Now $z \in A$ implies that z is polygonally connected to u. This implies that all points in $B(z;r)$ are polygonally connected to u. Hence $B(z;r) \subset A$. This implies that A is open.

Suppose $B \neq \phi$. Let $z' \in B$. Then there exists $r' > 0$ such that $B(z';r') \subset D$. Note that none of the points in $B(z';r')$ is polygonally connected to u, for otherwise, z' would be polygonally connected to u and z' would not be in B. Hence, $B(z';r') \subset B$ and B is open. But now D is connected and $D = A \cup B$ where A and B are open sets. Since A is non-empty, the only way this can happen is $B = \phi$. Hence $D = A$ and every point in D is polygonally connected to u and hence, the open set D is polygonally connected.

Conversely, suppose D is not connected. Then let A and B be open disjoint sets such that

$$D = A \cup B.$$

Let $a \in A$ and $b \in B$. Suppose that there exists a polygonal line connecting a to b. We may assume this line to be the line segment $[a, b]$. For if not, there is a line contained in the polygonal path that joins a point a^* in A with a point b^* in B for the first time. We then replace a by a^* and b by b^*.

Now, let $\gamma : [0, 1] \to D$ be $\gamma(t) = a(1-t) + bt$. Let

$$t^* = \sup\{t \in [0, 1] | \gamma(t) \in A\}.$$

Since A is open, $t^* > 0$. Similarly since B is open, $t^* < 1$. Let $z^* = \gamma(t^*)$. Note that $z^* \notin A$. For if $z^* \in A$, then since A is open, there exists $\epsilon > 0$ such that

$B(z^*;\epsilon) \subset A$. By continuity of γ, we conclude that there exists $\delta_\epsilon > 0$ such that if $|t - t^*| < \delta_\epsilon$ then $\gamma(t) \in B(z^*;\epsilon)$. But this means that $\gamma(t^* + \delta_\epsilon/2) \in A$ and t^* is not an upper bound for the set $\{t \in [0,1] | \gamma(t) \in A\}$.

Similarly, B does not contain z^*. For if B contains z^* then there exists $\epsilon' > 0$ such that if $|t - t^*| < \delta_{\epsilon'}$, then $\gamma(t) \in B(z^*;\epsilon') \subset B$. This implies that $\gamma(t) \in B$ for $t^* - \delta'_\epsilon < t$ and so, t^* is not the least upper bound for the set $\{t \in [0,1] | \gamma(t) \in A\}$.

We must therefore conclude that such a line segment does not exist. □

8 The Residue Theorem

8.1 Residues

When z_0 is an isolated singularity of f, there exist an R such that f is analytic on $0 < |z - z_0| < R$. Therefore $f(z)$ has a Laurent series expansion given by Theorem 6.3, namely,

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z - z_0)^{n+1}}\, dz, n = 0, 1, 2, \cdots ,$$

and

$$b_n = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)}{(z - z_0)^{-n+1}}\, dz, n = 1, 2, \cdots .$$

When $n = 1$,

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz,$$

and this is called the residue of $f(z)$ at the isolated singular point z_0. It is denoted by $\mathrm{Res}(f(z); z_0)$.

EXAMPLE 8.1 Let

$$f(z) = \frac{e^{-z}}{(z - 1)^2}.$$

Find $\mathrm{Res}(f(z), 1)$.

Solution

The function $f(z) = \dfrac{e^{-z}}{(z - 1)^2}$ is analytic in $|z| \leq 2$ except at the isolated singularity $z = 1$. By Cauchy Integral Formula,

$$\int_C \frac{e^{-z}}{(z - 1)^2} dz = -\frac{2\pi i}{e}.$$

Thus, the residue of $f(z)$ at $z = 1$ is

$$\text{Res}(f, 1) = -\frac{1}{e}.$$

EXAMPLE 8.2 Let $f(z) = e^{1/z^2}$. Find $\text{Res}(f(z), 0)$.

Solution

We have

$$e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \cdots.$$

The coefficient of $\frac{1}{z}$ is 0. Therefore

$$\int_C e^{1/z^2}\, dz = 0.$$

This implies $\text{Res}(f; 0) = 0$.

This example cannot be deduced from Cauchy's Integral Formula.

8.2 Residue Theorem

If a function f has only a finite number of singular points interior to a given simple closed contour C, they must be isolated. The following Theorem gives us a formula for evaluating

$$\int_C f(z)dz$$

if f has a finite number of singular points interior to C.

THEOREM 8.1 (The Residue Theorem) Let C be a positively oriented simple closed contour within and on which a function f is analytic except for a finite number of singular points $z_1, z_2, \cdots, z_n$ interior to C. If $B_1, B_2, \cdots, B_n$ denote the residues of f at these respective points, then

$$\int_C f(z)dz = 2\pi i(B_1 + B_2 + \cdots + B_n).$$

Proof

By Cauchy Goursat Theorem, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \cdots + \int_{C_n} f(z)dz.$$

Since

$$\int_{C_i} f(z)dz = 2\pi i B_i,$$

we immediately obtain the result.

□

EXAMPLE 8.3 Suppose C is the circle $|z| = 2$ described anticlockwise. Evaluate

$$\int_C \frac{5z-2}{z(z-1)}dz.$$

Solution

Note that $z = 0$ and $z = 1$ are the two singularities of the function

$$f(z) = \frac{5z-2}{z(z-1)}.$$

$$\frac{5z-2}{z(z-1)} = \frac{2}{z} + 3 + \cdots, \quad |z| < 1$$

and

$$\frac{5z-2}{z(z-1)} = \left(5 + \frac{3}{z-1}\right)(1 - (z-1) + (z-1)^2 + \cdots), \quad |z-1| < 1,$$

therefore, Residue of $f(z)$ at $z = 0$ is 2 and at $z = 1$ is 3. Hence,

$$\int_C f(z)dz = 2\pi i(2+3) = 10\pi i.$$

Alternatively one may use Cauchy Integral Formula.

THEOREM 8.2 If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C f(z)dz = 2\pi i \text{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right).$$

Proof
Let $C(0;R)$ be a circle with R large enough so that the circle enclosed C. Then

$$\int_C f(z)dz = \int_{C(0;R)} f(z)dz.$$

Write

$$\int_{C(0;R)} f(z)dz = \int_0^{2\pi} f(Re^{it})iRe^{it}dt.$$

Replacing t by $-s$, we find that

$$-\int_0^{-2\pi} f(Re^{-is})iRe^{-is}ds.$$

We now convert this line integral back to contour integral by letting $z = e^{is}/R$ and deduce that

$$-i\int_0^{-2\pi} f(R/e^{is}) \cdot \frac{R^2}{e^{2is}} \cdot \frac{e^{is}}{R} ds = -\int_{C'(0;1/R)} \frac{f(1/z)}{z^2} dz$$

where $C'(0,1/R)$ is the circle centered at 0 with radius $1/R$ traversed in clockwise direction. Now, this gives

$$\int_{C(0;R)} f(z)dz = \int_{C(0;1/R)} \frac{f(1/z)}{z^2} dz$$

where $C(0;1/R)$ is traversed in anti-clockwise direction and the proof is complete.

□

EXAMPLE 8.4 Use Theorem 8.2 to solve Example 8.3.

Solution
Let $f(z) = (5z-2)/(z(z-1))$. Then

$$\frac{1}{z^2} f(1/z) = \frac{5-2z}{z}\left(1 + z + z^2 + \cdots\right)$$

and so $\text{Res}(1/z^2 f(1/z), 0) = 5$ and hence the result.

8.3 Evaluations of improper integrals

An important application of the theory of residues is the evaluation of certain types of definite improper integral arising from real analysis.

In Calculus we encounter improper integral of continuous function $f(x)$ over semi infinite interval $x \geq 0$:

$$\int_0^\infty f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx.$$

When the limit on the right exists, the improper integral is said to converge and its value is the value of the limit. The improper integral $\int_{-\infty}^{\infty} f(x)dx$ is defined by

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty}\int_0^{R_2} f(x)dx.$$

When both integrals on the right hand side converges, we say that $\int_{-\infty}^{\infty} f(x)dx$ converges. It may happen that the integrals on the right side diverge but the limit

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

exists. In this case, we call

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

the Cauchy principal value of the integral of $\int_{-\infty}^{\infty} f(x)dx$ and write

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx.$$

Remark 8.1 The Cauchy principal value of the integral $\int_{-\infty}^{\infty} f(x)dx$ may exist without $\int_{-\infty}^{\infty} f(x)dx$ being defined. For example, if $f(x) = 2x/(1+x^2)$, $\int_{-\infty}^{\infty} f(x)dx$ is divergent but its principal value is 0. However, when $f(x)$ is an even function, i.e., $f(x) = f(-x)$, both P.V.$\int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx$ coincide.

In this section, we will use residue theorem to evaluate different type of integrals.

EXAMPLE 8.5 Evaluate

$$\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}dx.$$

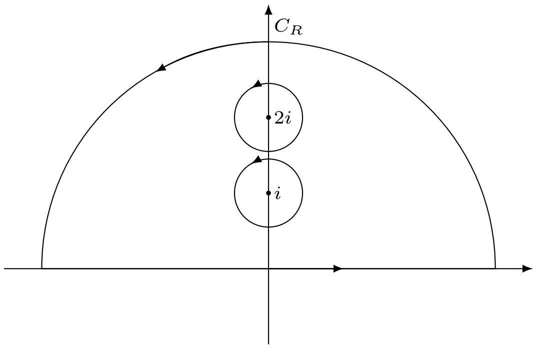
Since $f(x)$ is even it suffices to evaluate the Cauchy Principal value of the integral by Example 5.4.1. Consider the function

$$f(z) = \frac{2z^2-1}{z^4+5z^2+4}.$$

Note that

$$z^4 + 5z^2 + 4 = (z^2 + 4)(z^2 + 1).$$

Hence, $f(z)$ has poles at $\pm 2i$ and $\pm i$. Let $R > 2$.



From the Residue Theorem, we know that

$$\int_{C_R} f(z)dz = 2\pi i(\text{Res}(f(z); i) + \text{Res}(f(z); 2i)).$$

Now,

$$\text{Res}(f(z); i) = \frac{i}{2},$$

while

$$\text{Res}(f(z); 2i) = -\frac{3i}{4}.$$

Hence,

$$\int_{\Gamma_R} f(z)dz = \frac{\pi}{2}.$$

Let $\Gamma_R = [-R, R] \cup C_R$, where C_R is the arc from R to $-R$ and $[-R, R]$ is the line segment $[-R, R]$. Now, on C_R,

$$\left| \frac{2z^2 - 1}{z^4 + 5z^2 + 4} \right| \leq \frac{2R^2 + 1}{R^4 - 5R^2 - 4}.$$

Hence,

$$\left| \int_{C_R} f(z)dz \right| \leq \frac{\pi R(2R^2 + 1)}{R^4 - 5R^2 - 4},$$

which tends to 0 as $R \to \infty$. Therefore,

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0.$$

Hence,

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = \frac{\pi}{2}.$$

EXAMPLE 8.6 Show that

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

8.4 Improper Integrals involving cos

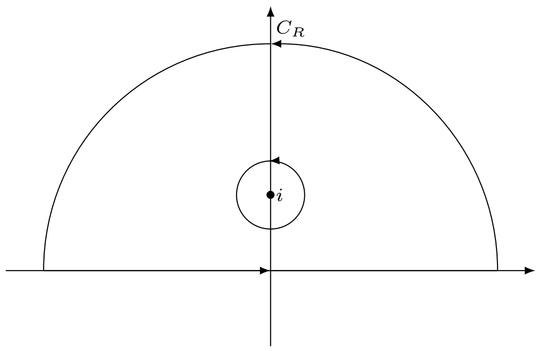
We now evaluate improper integrals of the type

$$\int_{-\infty}^\infty P(x)\cos x dx.$$

EXAMPLE 8.7 Show that

$$\int_{-\infty}^\infty \frac{\cos x}{(x^2+1)^2} dx = \frac{\pi}{e}.$$

Consider $f(z) = \dfrac{e^{iz}}{(z^2+1)^2}$.



Note that

$$\int_{C_R} f(z)dz = 2\pi i \mathrm{Res}(f(z); i).$$

Let

$$\frac{\phi(z)}{(z-i)^2} = \frac{e^{iz}}{(z-i)^2(z+i)^2},$$

or

$$\phi(z) = \frac{e^{iz}}{(z+i)^2}.$$

By Cauchy Integral formula,

$$\int_{C_R} \frac{e^{iz}}{(z-i)^2(z+i)^2}dz = \int_{C_R} \frac{\phi(z)}{(z-i)^2}dz = 2\pi i\phi'(i).$$

Now,

$$\phi'(z) = \frac{-2e^{iz}}{(z+i)^3} + \frac{ie^{iz}}{(z+i)^2}.$$

Hence

$$\phi'(i) = \frac{-i}{2e}.$$

Therefore,

$$\int_{\Gamma_R} f(z)dz = \frac{\pi}{e}.$$

Now split $\Gamma_R = [-R, R] \cup C_R$. Then on C_R

$$\left|\frac{e^{iz}}{(z^2+1)^2}\right| \leq \frac{1}{(R^2-1)^2},$$

which implies immediately that

$$\int_{C_R} f(z)dz \to 0$$

as $R \to \infty$.

Hence, we conclude that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}dx = \frac{\pi}{e}.$$

Splitting $e^{ix} = \cos x + i\sin x$, we deduce our result.

Remark 8.2 In the computation of the residue $\mathrm{Res}(f(z); i)$, we can also compute the Laurent series expansion of

$$\frac{1}{(z-i)^2}\frac{e^{iz}}{(z+i)^2} = \frac{1}{(z-i)^2}\frac{e^{-1}}{(2i)^2}\left(1+(z-i)i+\cdots\right)\left(1-2\frac{(z-i)}{2i}+\cdots\right).$$

The coefficient of $(z-i)^{-1}$ in this expansion is $(2ie)^{-1}$.

8.5 Euler's identities

The Bernoulli numbers B_m are defined as $f^{(m)}(0)$ where

$$f(z) = \frac{z}{e^z-1}.$$

In other words, we have

$$\frac{z}{e^z-1} = \sum_{m=0}^{\infty} B_m\frac{z^m}{m!}.$$

The first few values of B_m are

$$B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}$$
$$B_7 = 0, B_8 = -\frac{1}{30}, B_9 = 0, B_{10} = \frac{5}{66}, B_{11} = 0, B_{12} = -\frac{691}{2730}.$$

In this section, we will show that

$$\sum_{k=1}^{\infty} \frac{1}{k^{2m}} = \frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}}{2\cdot(2m)!}. \tag{8.1}$$

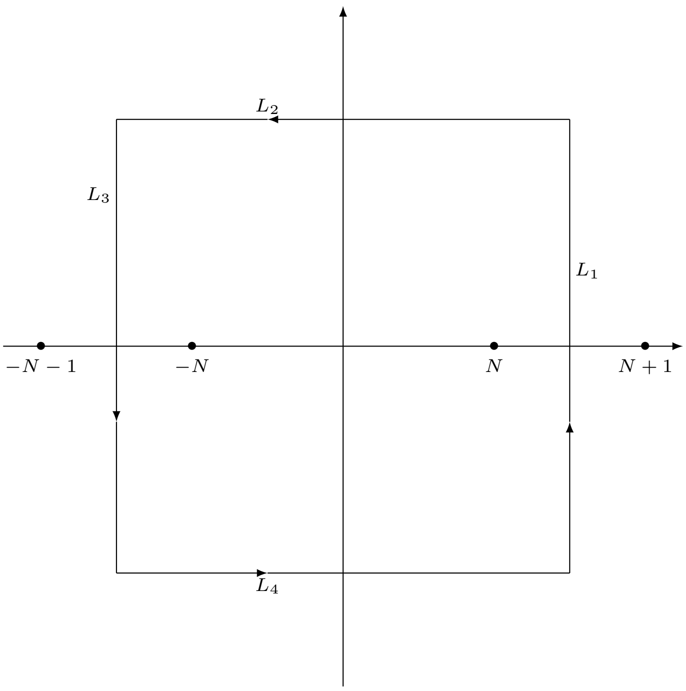
Using the values of B_2 and B_4, we deduce that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

We begin with the integral

$$\frac{1}{2\pi i}\int_{C_N} \frac{1}{\zeta^{2m+1}}\frac{2\pi i\zeta}{e^{2\pi i\zeta}-1}\,d\zeta,$$

where C_N is the following contour:



The function

$$f(z) = \frac{1}{z^{2m+1}}\frac{2\pi iz}{e^{2\pi iz}-1}$$

has poles $0, \pm k, k = 1, 2, \cdots, N$ which are enclosed by C_N. This implies that

$$\frac{1}{2\pi i}\int_{C_N} f(\zeta)d\zeta = \text{Res}(f(z);0) + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \text{Res}(f(z);k).$$

By using the Laurent series expansion of $f(z)$, we deduce that

$$\text{Res}(f(z);0) = \frac{B_{2m}(-1)^m(2\pi)^{2m}}{(2m)!}$$

and

$$\text{Res}(f(z);k) = \lim_{z\to k} f(z)(z-k) = \frac{1}{k^{2m}}.$$

To derive (8.1), it suffices to show that

$$\frac{1}{2\pi i}\int_{C_N} f(\zeta)d\zeta \to 0$$

as $N \to \infty$. We will only show that the integrals over L_1 and L_2 vanishes as $N \to \infty$. The computations for the integrals over L_3 and L_4 are similar.

We first give an upper bound for $\left|1 - e^{2\pi iz}\right|$ on L_1 and L_2. We first parametrize L_1 by $\zeta(t) = N + 1/2 + it, -(N+1/2) \le t \le N + 1/2$. Note that

$$\left|1 - e^{2\pi i(N+1/2+it)}\right| = \left|1 + e^{-2\pi t}\right| \ge 1$$

and this gives the lower bound of $\left|1 - e^{2\pi iz}\right|$ on L_1.

On L_2, we use the parametrization $\zeta(t) = t + i(N+1/2), -(N+1/2) \le t \le N + 1/2$. Note that

$$\left|1 - e^{2\pi i(t+(N+1/2)i)}\right| \ge 1 - e^{-2\pi(N+1/2)}.$$

Write

$$e^{-2\pi N}e^{-\pi} \le e^{-\pi}$$

and hence

$$1 - e^{-2\pi(N+1/2)} \ge 1 - e^{-\pi} > 0.$$

This gives us the lower bound of $\left|1 - e^{2\pi iz}\right|$ on L_2.

With these lower bounds, we deduce that

$$\left|\int_{L_1} f(\zeta)d\zeta\right| \le \frac{2N+1}{(N+1/2)^{2m+1}}2\pi(N+1/2)$$

and

$$\left|\int_{L_2} f(\zeta)d\zeta\right| \le \frac{2N+1}{(N+1/2)^{2m+1}}\frac{2\pi(N+1/2)}{1-e^{-\pi}}$$

and this implies that both integrals over L_1 and L_2 vanishes as $N \to \infty$. This completes the proof of (8.1).

8.6 Residue Theorem and identities associated with binomial coefficients

In combinatorics, we often encounter combinatorial identities. In the next two examples, we show how such identities can be derived using residue theorem.

EXAMPLE 8.8 Show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Solution

We observe that

$$\binom{2n}{n} = \frac{1}{2\pi i} \int_{C(0;1)} \frac{(1+\zeta)^{2n}}{\zeta^{n+1}}\, d\zeta.$$

The integral on the right hand side can be written as

$$\frac{1}{2\pi i} \int_{C(0;1)} \frac{(1+\zeta)^n \left(1+\dfrac{1}{\zeta}\right)^n}{\zeta}\, d\zeta = \frac{1}{2\pi i} \int_{C(0;1)} \frac{1}{\zeta} \left(\sum_{k=0}^{n} \sum_{\ell=0}^{n} \binom{n}{k} \binom{n}{\ell} \zeta^{k-\ell} \right) d\zeta$$
$$= \sum_{k=0}^{n} \binom{n}{k}^2$$

since

$$\frac{1}{2\pi i} \int_{C(0;1)} \zeta^{n-\ell-1}\, d\zeta = 1$$

if and only if $n - \ell - 1 = -1$ or $n = \ell$.

EXAMPLE 8.9 Evaluate

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}.$$

Solution

Observe that

$$\binom{2n}{n} \frac{1}{5^n} = \frac{1}{2\pi i} \int_C \frac{(1+z)^{2n}}{5^n z^{n+1}} dz$$

where C is the circle $C(0; r)$ traversed in the counterclockwise direction and

$r > 0$ to be chosen later. This implies that

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} = \frac{1}{2\pi i} \int_{C(0;r)} \frac{1}{z} \sum_{n=0}^{\infty} \frac{(1+z)^{2n}}{(5z)^n} dz = \frac{1}{2\pi i} \int_{C(0;r)} \frac{5}{3z - 1 - z^2} dz,$$

where r can be chosen to be 1 to ensure that on $|z| = 1$,

$$\left| \frac{(1+z)^2}{(5z)} \right| \leq \frac{4}{5}$$

and that the use of the expansion

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$$

is valid. The function $3z - 1 - z^2$ has zeroes $\frac{3}{2} \pm \frac{\sqrt{5}}{2}$ and only $\frac{3}{2} - \frac{\sqrt{5}}{2}$ is enclosed by $C(0;r)$. Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} &= \frac{1}{2\pi i} \int_{C(0;r)} \frac{1}{z} \sum_{n=0}^{\infty} \frac{(1+z)^{2n}}{(5z)^n} dz \\
&= \frac{1}{2\pi i} \int_{C(0;r)} \frac{5}{3z - 1 - z^2} dz \\
&= \operatorname{Res} \left(-\frac{5}{z^2 + 1 - 3z}, \frac{3 - \sqrt{5}}{2} \right) = \sqrt{5}.
\end{aligned}$$

8.7 An improper integral involving $\sin x$

We have encounter integrals that involve $\cos x$. In this section, we will show the evaluations of a improper integral that involve $\sin x$.

EXAMPLE 8.10 Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx.$$

Solution

We first establish a useful inequality called the Jordan inequality given by

$$\int_0^{\pi} e^{-R \sin \theta} \, d\theta < \frac{\pi}{R}, \tag{8.2}$$

where R is a real positive number. To prove (8.2), we note that for $0 \leq \theta \leq \pi/2$,

$$\sin \theta \geq \frac{2}{\pi} \theta.$$

This follows immediately from the observation that for $0 \le \theta \le \pi/2$ the graph of $y = 2\theta/\pi$ lies below the graph of $y = \sin\theta$. Therefore,

$$\int_0^{\pi/2} e^{-R\sin\theta}\,\theta \le \int_0^{\pi/2} e^{-2\theta R/\pi}\,d\theta = \frac{\pi}{2R} - \frac{\pi e^{-R}}{2R} < \frac{\pi}{2R}.$$

Similarly, by observing that for $\pi/2 \le \theta \le \pi$,

$$\sin\theta \ge -\frac{2}{\pi}\theta + 2,$$

we conclude that

$$\int_{\pi/2}^{\pi} e^{-R\sin\theta}\,\theta < \frac{\pi}{2R}.$$

Using the bounds for the two integrals we have just discussed, we complete the proof of (8.2).

We are now ready to evaluate our contour in this example. Consider the contour similar to Example 8.5. Note that

$$\int_{\Gamma_R} \frac{e^{iz}-1}{z}dz = 0.$$

Hence,

$$\int_{C_R} \frac{e^{iz}-1}{z}dz + \int_{-R}^{R} \frac{e^{ix}-1}{x}dx = 0.$$

Now,

$$\int_{-R}^{R} \frac{e^{ix}-1}{x}dx = -\int_{C_R} \frac{e^{iz}-1}{z}dz = \pi i + \int_{C_R} \frac{e^{iz}}{z}dz.$$

Now,

$$\left|\int_{C_R} \frac{e^{iz}}{z}dz\right| \le \int_0^{\pi} e^{-R\sin\theta}d\theta < \frac{\pi}{R},$$

where the last inequality follows from (8.2). Therefore,

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}dz = 0.$$

This implies that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}dx = \lim_{R\to\infty} \operatorname{Im} \int_{-R}^{R} \frac{e^{ix}-1}{x}dx = \pi.$$

9 Winding Number

9.1 Winding number and Cauchy's Residue Theorem for closed curves

We have so far discussed only integral over simple closed curve. It is more natural to consider general closed curve, i.e., curve that intersects itself several times.

Consider the function $f(z) = z^m$, with $m \geq 2$ being a positive integer. Then the integral

$$\int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta$$

can be written as

$$\int_{f(\gamma)} \frac{1}{\omega} d\omega,$$

where γ is $C(0;1)$ and $f(\gamma)$ is the image of the curve γ under f. Note that γ can be parametrized by $z(t) = e^{it}, 0 \leq t \leq 2\pi$. When t increases from 0 to $2\pi/m$, we see that z^m moves around the origin in one full circle. By the time t reaches 2π, z^m would have circled the origin m times. In other words, the curve $f(\gamma)$ "winds" around the origin m times and we observe that

$$\frac{1}{2\pi i} \int_{f(\gamma)} \frac{1}{\omega} d\omega = \frac{1}{2\pi i} \int_\gamma \frac{(z^m)'}{z^m} dz = m. \tag{9.1}$$

This motivates the following definition:

DEFINITION 9.1 Suppose γ is a closed curve (not necessarily simple) and that $a \notin \gamma$. Then

$$n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a} dz$$

is called the winding number of γ around a.

THEOREM 9.1 For any closed curve γ and $a \notin \gamma$, $n(\gamma; a)$ is an integer.

Proof
Let γ be parametrized by $z(t)$, $0 \le t \le 1$. Set

$$F(s) = \int_0^s \frac{z'(t)}{z(t) - a} dt, 0 \le s \le 1.$$

Let

$$G(s) = \frac{e^{F(s)}}{z(s) - a}.$$

Now, since

$$F'(s) = \frac{z'(s)}{z(s) - a},$$

we conclude that

$$G'(s) = e^{F(s)} \frac{F'(s)}{z(s) - a} - \frac{e^{F(s)} z'(s)}{(z(s) - a)^2} = 0.$$

Hence,

$$G(s) = \frac{e^{F(s)}}{z(s) - a} = C,$$

where C is a constant. Let $s = 0$. Since $F(0) = 0$, we find that

$$C = G(0) = \frac{1}{z(0) - a}. \tag{9.2}$$

When $s = 1$, we find that

$$C = G(1) = \frac{e^{F(1)}}{z(1) - a}. \tag{9.3}$$

Now, γ is a closed curve and this implies that $z(0) = z(1)$ and we deduce from (9.2) and (9.3) that

$$e^{F(1)} = 1$$

or

$$F(1) = 2\pi i n(\gamma; a) = 2\pi i k, k \in \mathbf{Z}.$$

In other words, we have shown that $n(\gamma; a)$ is an integer. □

THEOREM 9.2 Suppose f is analytic in a star-shaped domain except for singularities at $z_1, z_2, \cdots, z_m$. Let γ be a closed curve enclosing these singularities. Then

$$\int_\gamma f(\zeta) d\zeta = 2\pi i \sum_{k=1}^{m} n(\gamma; z_k) \mathrm{Res}(f, z_k).$$

Proof
We know that around each z_j, $f(z)$ can be written as

$$f(z) = \sum_{k=0}^{\infty} a_{j,k}(z - z_j)^k + \sum_{k=1}^{\infty} \frac{b_{j,k}}{(z - z_j)^k}.$$

The expression

$$\sum_{k=1}^{\infty} \frac{b_{j,k}}{(z - z_j)^k}$$

is called the principal part of the Laurent series expansion at z_j. Collecting all the principal parts of the Laurent series expansion of $f(z)$ at each $z_j, 1 \leq j \leq m$, we may write

$$f(z) = \sum_{j=1}^{m} \sum_{k=1}^{\infty} \frac{b_{j,k}}{(z - z_j)^k} + h(z),$$

where now $h(z)$ is analytic in the region bounded by γ. By Cauchy-Goursat Theorem,

$$\int_\gamma h(\zeta) d\zeta = 0.$$

Therefore,

$$\begin{aligned} \int_\gamma f(\zeta) d\zeta &= \sum_{j=1}^{m} \int_\gamma \sum_{k=1}^{\infty} \frac{b_{j,k}}{(\zeta - z_j)^k} d\zeta \\ &= \sum_{j=1}^{m} \int_\gamma \frac{b_{j,1}}{\zeta - z_j} d\zeta = 2\pi i \sum_{j=1}^{m} n(\gamma; z_j) \text{Res}(f; z_j), \end{aligned}$$

since $b_{j,1} = \text{Res}(f; z_j)$ and $\frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z_j} d\zeta = n(\gamma; z_j)$. □

Remark 9.1 In the proof above, we use Laurent series expansions of $f(z)$ at various singularities and extract the residues from these expansions.

9.2 Counting zeroes and poles

THEOREM 9.3 Suppose γ is a simple closed curve. If f is meromorphic (meaning that the singularities of f are poles) inside and on γ and the zeroes and poles of f are not on γ, then

$$Z_\gamma(f) - P_\gamma(f) = \frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta,$$

where $Z_\gamma(f)$ is the number of zeroes of f enclosed by γ (a zero of order k being counted k times) and $P_\gamma(f)$ is the number of poles of f enclosed by γ (a pole of order ℓ being counted ℓ times).

Proof
Let $z_1, z_2, \cdots, z_m$ be distinct zeroes of f with multiplicity $\alpha_1, \alpha_2, \cdots, \alpha_m$ and $p_1, p_2, \cdots, p_\ell$ be distinct poles of f with multiplicity $\beta_1, \beta_2, \cdots, \beta_\ell$. Note that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta = \sum_{u=1}^{m} \text{Res}\left(\frac{f'(z)}{f(z)}; z_u\right) + \sum_{v=1}^{\ell} \text{Res}\left(\frac{f'(z)}{f(z)}; p_v\right).$$

If a were a zero of f with multiplicity s, then

$$f(z) = (z-a)^s g(z)$$

with $g(a) \neq 0$. This implies that

$$\frac{f'(z)}{f(z)} = \frac{s}{z-a} + \frac{g'(z)}{g(z)}$$

and the residue of f'/f at a is s.

Similarly if b were a pole of f with multiplicity t, then

$$f(z) = \frac{h(z)}{(z-b)^t}$$

with $h(z)$ analytic at b. This implies that

$$\frac{f'(z)}{f(z)} = \frac{-t}{z-a} + \frac{h'(z)}{h(z)}$$

and the residue of f'/f at b is $-t$. Hence, the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta = Z_\gamma(f) - P_\gamma(f).$$

□

If f has no poles in the region enclosed by γ, then we have

$$Z_\gamma(f) = \frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta.$$

THEOREM 9.4 (Rouché's Theorem) Suppose f and g are analytic inside and on a simple closed curve γ and that $|f(z)| > |g(z)|$ for all $z \in \gamma$. Then

$$Z_\gamma(f+g) = Z_\gamma(f)$$

inside γ where $Z_\gamma(h)$ is the number of zeroes of $h(z)$ enclosed by γ.

Proof
Note that by writing

$$f+g=f\left(1+\frac{g}{f}\right)$$

and noting that

$$\frac{(f+g)'}{f+g}=\frac{f'}{f}+\frac{(1+g/f)'}{1+g/f},$$

we conclude that

$$Z_\gamma(f+g)=Z_\gamma(f)+\frac{1}{2\pi i}\int_\gamma\frac{1+(g(\zeta)/f(\zeta))'}{1+g(\zeta)/f(\zeta)}d\zeta.$$

But the last integral can be written as

$$\int_{F(\gamma)}\frac{1}{\omega}d\omega,$$

where

$$\omega=F(z)=1+\frac{g(z)}{f(z)}.$$

Now,

$$|f|>|g|$$

implies that

$$|g/f|<1 \quad \text{or} \quad |g/f+1-1|<1.$$

Hence, $F(\gamma)\subset B(1;1)$ and the function $1/\omega$ is analytic in $B(1;1)$. Therefore, the last integral is 0 and we have

$$Z_\gamma(f)=Z_\gamma(f+g).$$

□

EXAMPLE 9.1 Find the number of zeroes of $e^z/3-z$ in $B(0;1)$.

Solution
Observe that on $|z|=1$,

$$|e^z/3|=e^{\mathrm{Re}(z)}/3\le e^{|z|}/3=e/3<1=|z|.$$

By Rouché's Theorem,

$$Z_{C(0;1)}(-z)=Z_{C(0;1)}(e^z/3-z).$$

Since $Z_{C(0;1)}(-z)=1$ (0 is the zero of $-z$ in $B(0;1)$), we conclude that

$$Z_{C(0;1)}(e^z/3-z)=1.$$

EXAMPLE 9.2 Let $\lambda > 1$. Show that $z + e^{-z} - \lambda = 0$ has exactly one solution in the region $\mathrm{Re}(z) > 0$.

Solution

The region $\mathrm{Re}(z) > 0$ is unbounded. But we will consider the contour

$$\gamma_R = [-iR, iR] + C_R$$

where C_R is the semi-circle of radius $R > 0$ which lies on the right half plane. On $[-iR, iR]$,

$$|e^{-z}| = |e^{-i\alpha}| = 1 < |i\alpha - \lambda| = |z - \lambda| = |\lambda - z|,$$

since

$$|i\alpha - \lambda| \geq \lambda > 1.$$

On C_R,

$$|e^{-z}| = e^{-\mathrm{Re}(z)} < e^0 = 1 < R - \lambda,$$

whenever $R > \lambda + 1$. Now, on C_R, we may write

$$R - \lambda = |z| - \lambda \leq |z - \lambda|.$$

Therefore

$$|e^{-z}| < |\lambda - z|$$

on C_R. Combining with the estimate on $[-iR, iR]$, we conclude that

$$|e^{-z}| < |\lambda - z|$$

on γ_R. This implies that

$$1 = Z_{\gamma_R}(-z + \lambda) = Z_{\gamma_R}(e^{-z} - z + \lambda)$$

and proves the claim in the problem.

9.3 Open mapping Theorem

In this section, we will use Rouché's Theorem to prove the open mapping theorem. We will follow the proof given in the book "Complex Analysis" by Stein and Shakarchi.

THEOREM 9.5 The image of an open set under a nonconstant analytic mapping is an open set.

Proof
Let V be an open set. We want to show that $f(V)$ is open. Let $w_0 \in f(V)$. Then there exists a $z_0 \in V$ such that $f(z_0) = w_0$. Since V is open, there exists $\delta > 0$ such that $B(z_0;\delta) \subset V$. Choose $\eta > 0$ such that $\overline{B(z_0;\eta)} \subset B(z_0;\delta)$ and the boundary $C(z_0;\eta)$ does not pass through the zeroes of $f(z) - w_0$, which is possible by Uniqueness Theorem. Since $f(z) - w_0 \neq 0$ on $C(z_0;\eta)$, we observe that

$$\epsilon := \min_{z \in C(z_0;\eta)} |f(z) - w_0| > 0.$$

For all $z \in C(z_0;\eta)$,

$$|f(z) - w_0| \geq \epsilon.$$

Now suppose that $|w - w_0| < \epsilon$. then

$$f(z) - w_0| \geq \epsilon > |w - w_0| = |w_0 - w|.$$

By Rouché's Theorem, we conclude that

$$Z_{C(z_0;\eta)}(f(z) - w_0) = Z_{C(z_0;\eta)}(f(z) - w_0 + w_0 - w) = Z_{C(z_0;\eta)}(f(z) - w).$$

This means that $f(z) - w$ has at least a zero in $B(z_0;\eta)$. In other words, $w \in f(B(z_0;\eta)$ for each $|w - w_0| < \epsilon$. This implies that

$$B(w_0;\epsilon) \subset f(B(z_0;\eta)) \subset f(V).$$

Hence $f(V)$ is open. □

We can prove maximum modulus theorem for closed balls directly from the open mapping theorem. Let $f(z)$ be analytic on $B(z_0;r)$ and continuous on $\overline{B(z_0;r)}$. Suppose $|f(a)|$ is maximum for $a \in B(z_0;r)$. Now, $f(a) \in f(B(z_0;r)$, which is open by open opening theorem. This means that $B(f(a);s) \subset f(B(z_0;r))$ for some $s > 0$. This means that there exists a point ξ on the boundary of $B(f(a);s)$ such that $\xi = f(b)$ and $|f(b)| > |f(a)|$, which is a contradiction. (One can choose $\xi = f(a) + se^{i\theta}$ where $\theta = \arg f(a)$.)